Reasoning Ledgers

Strive for rigorous reasoning and gapless documentation.

Problems	Activity category	Scaffold	Pg.	Related topics	References
Algebra problems Geometry proofs Explanation problems	**Develop arguments**	**STEP**	2	formal arguments	Toulmin 2003 doi.org/10.1017/CBO9780511840005 McNeill, Lizotte, Krajcik, and Marx 2006 doi.org/10.1207/s15327809jls1502_1
				chains of reasoning	Speirs, Ferm, Stetzer, and Lindsey 2016 doi.org/10.1119/perc.2016.pr.077
	Narrate arguments		6, 9	(Same as for developing arguments)	
Word problems Textbook readings	**Read & write** notes & questions	**SCAN**	7	multiple representations	Larkin, McDermott, Simon, and Simon 1980 doi.org/10.1126/science.208.4450.1335 van Heuvelen 1991 doi.org/10.1119/1.16668
				idea distinct from name	Arons 1984 doi.org/10.1119/1.2341444
Physics and chemistry problems	**Spider-web thinking** (track quantities and governing and constitutive relationships that simultaneously constrain them)	**MAP**	8	novice and expert problem-solving, production schemas	Chi, Feltovich, and Glaser 1981 doi.org/10.1207/s15516709cog0502_2
				Simultaneously diagramming governing and constitutive relationships	Rosengrant 2011 doi.org/10.1119/1.3527754
Testing experiments in ACT® Science and AP® Physics	**Inquire** (reject candidate models)	**STEAM**	10	inquiry-based learning	Etkina *et al.* islephysics.net
				Modeling Instruction™	AMTA modelinginstruction.org
				patternicity	Shermer 2008 doi.org/10.1038/scientificamerican1208-48
Precalculus problems in classes with playful torture	Keep track of excluded values and **domain restrictions**	**PAINFUL**	11	implied domain	Jones people.richland.edu/james/lecture/m116/functions/functions.html

See pg. 12 for exercises.

Example 1.1: Solve for x.Show each step that can't be accomplished on a basic four-function calculator.

$$4x - 2 = 98$$

	Statement	**Tool**	**Equivalent parts** (of statement & tool)	**Populated tool / Point(s)**
1.	$4x - 2 = 98$	$a = b$ $\updownarrow$ $a + c = b + c$	$a = 4x - 2$ $b = 98$ $c = 2$	$4x - 2 = 98$ $\updownarrow$ $4x - 2 + 2 = 98 + 2$
2.	$4x + 0 = 100$	$a \leftrightarrow a + 0 \leftrightarrow 0 + a$	$a = 4x$	$4x \leftrightarrow 4x + 0 \leftrightarrow 0 + 4x$
3.	$4x = 100$	$a = b$ $\updownarrow$ $\frac{a}{c} = \frac{b}{c}, c \neq 0$	$a = 4x$ $b = 100$ $c = 4$	$4x = 100$ $\updownarrow$ $\frac{4x}{4} = \frac{100}{4}, 4 \neq 0$
4.	$\frac{4x}{4} = 25$	$a \leftrightarrow a \cdot 1 \leftrightarrow 1 \cdot a$	$a = 4$	$4 \leftrightarrow 4 \cdot 1 \leftrightarrow 1 \cdot 4$
5.	$\frac{4x}{4 \cdot 1} = 25$	$\frac{ab}{cd} \leftrightarrow \frac{a}{c}\frac{b}{d}$	$a = 4$ $b = x$ $c = 4$ $d = 1$	$\frac{4x}{4 \cdot 1} \leftrightarrow \frac{4}{4}\frac{x}{1}$
6.	$\frac{4}{4}\frac{x}{1} = 25$	$\frac{a}{a}, a \neq 0 \leftrightarrow 1$	$a = 4$	$\frac{4}{4}, 4 \neq 0 \leftrightarrow 1$
7.	$1 \cdot \frac{x}{1} = 25$	$\frac{a}{1} \leftrightarrow a$	$a = x$	$\frac{x}{1} \leftrightarrow x$
8.	$1 \cdot x = 25$	$a \cdot 1 \leftrightarrow 1 \cdot a \leftrightarrow a$	$a = x$	$x \cdot 1 \leftrightarrow 1 \cdot x \leftrightarrow x$
9.	$x = 25$			

To fill in the **Tool** column, use the basic algebra tools or basic pre-algebra tools on the following pages.

A two-column Statement-Reason proof in geometry only needs to show the **Statement** and **Tool** columns of a STEP table. Homework and tests in Algebra 1 and 2 classes typically only require the work in the **Statement** column to be shown.

Basic algebra tools

For exercises, see corresponding "EA" sections in Marecek *et al.*, *Elementary Algebra 2e*, available for free through a CC BY license at OpenStax: openstax.org/details/books/elementary-algebra-2e

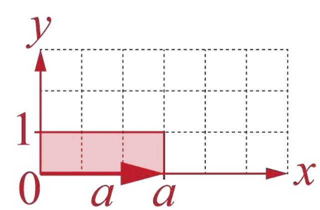
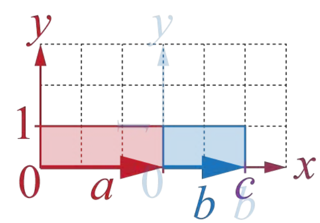
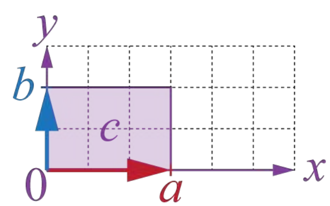
EA	Name	Tool(s)
1.9	**A.** Division by zero is undefined	Any denominator or factor in denominator = 0 ↓ Discard template
1.5	**B.** Product of fractions	$\frac{a}{b}\cdot\frac{c}{d} \leftrightarrow \frac{ac}{bd}$
1.5	**C.** Canceling factors in numerator & denominator	$\frac{a\cdot c}{b\cdot c} \leftrightarrow \frac{a}{b}$; $\frac{c\cdot a}{c\cdot b} \leftrightarrow \frac{a}{b}$
1.5	**D.** Quotient of fractions	$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} \leftrightarrow \frac{a}{b}\cdot\frac{d}{c}$
1.9	**E.** Commutative Property of Addition	$a+b$ $\updownarrow$ $b+a$
1.9	**F.** Commutative Property of Multiplication	$a\cdot b$ $\updownarrow$ $b\cdot a$
1.9	**G.** Associative Property of Addition	$a+b+c$ $\updownarrow$ $(a+b)+c$ $\updownarrow$ $a+(b+c)$
1.9	**H.** Associative Property of Multiplication	$a\cdot b\cdot c$ $\updownarrow$ $(a\cdot b)\cdot c$ $\updownarrow$ $a\cdot(b\cdot c)$

EA	Name	Tool(s)
1.9	**I.** Distributive Property	$a\cdot(b+c) \updownarrow a\cdot b+a\cdot c$; $(b+c)\cdot a \updownarrow b\cdot a+c\cdot a$ $a\cdot(b-c) \updownarrow a\cdot b-a\cdot c$; $(b-c)\cdot a \updownarrow b\cdot a-c\cdot a$ $(a+b)\cdot(c+d)$ $\updownarrow$ $\begin{array}{c\|c\|c} & c & d \\ \hline a & a\cdot c & a\cdot d \\ \hline b & b\cdot c & b\cdot d \end{array}$ $a\cdot c+b\cdot c+a\cdot d+b\cdot d$
1.9	**J.** Identity Property of Addition	$a+0 \leftrightarrow 0+a \leftrightarrow a$
1.9	**K.** Identity Property of Multiplication	$a\cdot 1 \leftrightarrow 1\cdot a \leftrightarrow a$
1.9	**L.** Identity Property of Division	$\frac{a}{1} \leftrightarrow a$
1.9	**M.** Adding additive inverse is equivalent to subtracting	$a+(-b)$ $\updownarrow$ $a-b$
1.9	**N.** Inverse Property of Addition	$a+(-a) \leftrightarrow a-a \leftrightarrow 0$
1.5 & 1.9	**O.** Inverse Property of Multiplication (and the related One Property of Division)	$a\cdot\frac{1}{a}, a\neq 0 \leftrightarrow \frac{a}{a}, a\neq 0 \leftrightarrow 1$

EA	Name	Tool(s)	
1.9	**P.** Multiplication by zero	$a \cdot 0 \leftrightarrow 0 \cdot a \leftrightarrow 0$	
1.9	**Q.** Division involving zero	$\frac{0}{a} \leftrightarrow 0$	See "Division by zero is undefined"
2.1	**R.** Addition Property of Equality	$a = b$ $\downarrow$ $\begin{array}{ccc} a & = & b \\ +c & & +c \\ \hline a+c & = & b+c \end{array}$	
2.1	**S.** Subtraction Property of Equality	$a = b$ $\downarrow$ $\begin{array}{ccc} a & = & b \\ -c & & -c \\ \hline a-c & = & b-c \end{array}$	
2.2	**T.** Multiplication Property of Equality	$a = b$ $\downarrow$ $a \cdot c = b \cdot c$	
2.2	**U.** Division Property of Equality	$a = b$ $\downarrow$ $\frac{a}{c} = \frac{b}{c}$	

EA	Name	Tool(s)	
6.2	**V.** Exponent notation	$a^2 \leftrightarrow a \cdot a$	$a^n \leftrightarrow \underbrace{a \cdot a \cdots a}_{n \text{ copies}}$
7.6	**W.** Zero-Product Property	$a \cdot b = 0$ $\downarrow$ $a = 0$ or $b = 0$	
9.1/9.7	**X.** Root notation	$\sqrt[n]{a} = b$, even n $\updownarrow$ $b^n = a, b \geq 0$, even n	$\sqrt[n]{a} = b$, odd n $\updownarrow$ $b^n = a$, odd n
9.8*	**Y.** A correction to the Power Property	$\sqrt{a^2} \leftrightarrow \lvert a \rvert$	
10.3	**Z.** Quadratic Formula	$ax^2 + bx + c = 0$ $\updownarrow$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	

BASIC PRE-ALGEBRA TOOLS

	Name	Numeric example	Tool: Algebraic	Tool: Gridded	Gridding steps
A.	**(Real) number**	3	a		1. Draw horizontal arrow from origin (0) to x-tick labeled a. Label arrow a. 2. Regard horizontal arrow as bottom edge of a rectangle of height 1. 3. Horizontal arrow and rectangle represent the number a individually and together.
B.	**Addition**	$3 + 2 = 5$	$a + b = c$		1. Represent the number a. 2. Regard arrowhead of arrow for a as origin of a new axis system. 3. Use new axis system to represent the number b. 4. Use original axes to read off x-tick, c, of arrowhead you just drew and complete sketch of the number c.
C.	**Subtraction**	$2 = 5 - 3$	$b = c - a$		
E.	**Multiplication**	$(3)(2) = 6$	$(a)(b) = c$		1. Represent number a. Erase rectangle. 2. Draw vertical arrow from origin (0) to y-tick labeled b. 3. Draw rectangle using horizontal arrow for a and vertical arrow for b as sides. Label signed number of unit squares, c, rectangle covers. 4. A rectangle in Quadrant I or Quadrant III has a positive signed number of unit squares. A rectangle in Quadrant II or Quadrant IV has a negative signed number of unit squares.
F.	**Division**	$2 = \frac{6}{3}$	$b = \frac{c}{a}, a \neq 0$		

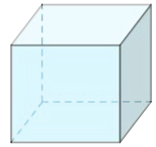
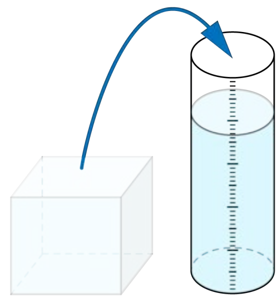
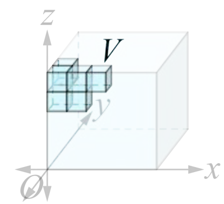
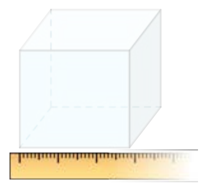
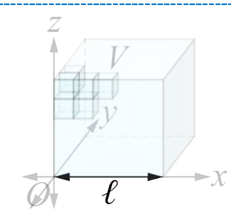
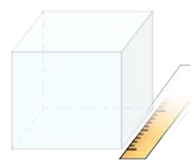
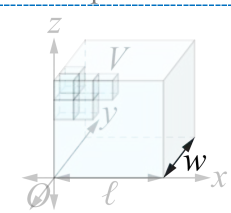
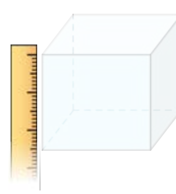
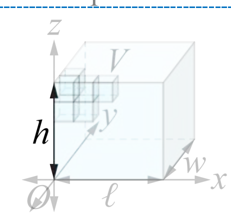
LESSON 2: Narrate algebra using STEP.

Example 2.1: Narrate row 1 in the STEP table from Example 1.1.

	Style	Skip statement	Tool	Equivalent parts (of statement & tool)	Populated tool / Point(s)
1.	Algebraic	When asked to write an explanation, you often need not rewrite the problem statement or current line of work, which is typically already nearby.	By the Addition Property of Equality, $a = b$ implies $a + c = b + c$.	Regard $4x - 2$ as a, 98 as b, and 2 as c.	So, $4x - 2 = 98$ implies $4x - 2 + 2 = 98 + 2$.
2.	Natural		By the Addition Property of Equality, a common quantity can be added to both sides of a true equation.	So, in the equation setting left side $4x - 2$ equal to right side 98, adding 2 to both sides	gives $4x - 2 + 2 = 98 + 2$.
3.	Brief (merged parts)		By the Addition Property of Equality, a common quantity, say 2, can be added to both sides of a true equation, say $4x - 2 = 98$. So $4x - 2 + 2 = 98 + 2$.		
4.	Even shorter (omit name of tool)		A common quantity, say 2, can be added to both sides of a true equation, say $4x - 2 = 98$. So $4x - 2 + 2 = 98 + 2$.		

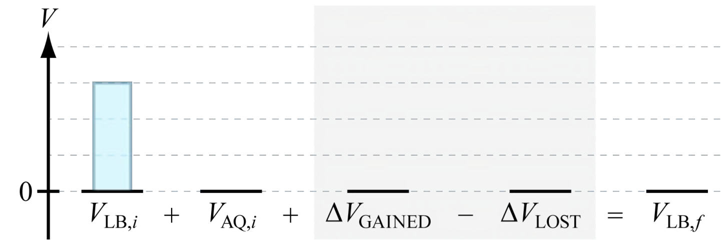
LESSON 3: **Read and write** notes and questions using **SCAN**.

Example 3.1: Translate: "A rectangular parcel of water has a volume equaling the parcel's length times the parcel's width times the parcel's height."

	Short phrase	**Cartoon**	**Axes and plots**	**Notation**
1.	A rectangular parcel of water	Semi-transparent rectangular prism 		
2.	has a volume	Arrow indicating water can be poured from rectangular container into graduated cylinder used to measure volume 	Cartesian axis system implying scale for representative unit cubes within parcel 	Let V = Volume
3.	equaling		Optional: Number line with two arrows pointing from the origin to same tickmark	$V =$
4.	the parcel's length	Optional: Ruler used to measure length 	Double-headed arrow indicating size of an edge parallel to x-axis 	Let ℓ =width $V = \ell$
5.	times		Optional: Area model of multiplication	$V = \ell \cdot$
6.	the parcel's width	Optional: Ruler used to measure width 	Double-headed arrow indicating size of an edge parallel to y-axis 	Let w =width $V = \ell \cdot w$
7.	times		Optional: Area model of multiplication	$V = \ell \cdot w \cdot$
8.	the parcel's height.	Optional: Ruler used to measure height 	Double-headed arrow indicating size of an edge parallel to z-axis 	Let h = height $V = \ell \cdot w \cdot h$

Example 4.1: Make a MAP for the following problem:

A rectangular lunchbox full of water is emptied completely into an initially empty rectangular aquarium that's longer, wider, and taller than the lunchbox. Is the height of the water in the aquarium greater than, less than, or the same as the height of the water initially in the lunchbox?



	Engineer's jargon		Name	Notation	Preferred layout
Managers	**Governing relationships** among properties belonging to various objects in the system or to interactions with the environment	**1.**	Volume conservation for incompressible fluid	$V_{\text{TOT,BEGIN}} + \Delta V_{\text{GAINED}} - \Delta V_{\text{LOST}} = V_{\text{TOT,END}}$	Near center/top Horizontal
Associated Personnel	**Constitutive relationships** among properties belonging to an individual object within a system or to an interaction with the environment	**2.**	Initial volume of rectangular parcel of water in lunchbox	$V_{\text{LB},i} = \ell_{\text{LB},i} \cdot w_{\text{LB},i} \cdot h_{\text{LB},i}$	Near bottom/edges Vertical
		3.	Initial volume of rectangular parcel of water in aquarium	$V_{\text{AQ},i} = \ell_{\text{AQ},i} \cdot w_{\text{AQ},i} \cdot h_{\text{AQ},i}$	
		4.	Final volume of rectangular parcel of water in lunchbox	$V_{\text{LB},f} = \ell_{\text{LB},f} \cdot w_{\text{LB},f} \cdot h_{\text{LB},f}$	
		5.	Final volume of rectangular parcel of water in aquarium	$V_{\text{AQ},f} = \ell_{\text{AQ},f} \cdot w_{\text{AQ},f} \cdot h_{\text{AQ},f}$	

$$V_{\text{TOT,BEGIN}} + \Delta V_{\text{GAINED}} - \Delta V_{\text{LOST}} = V_{\text{TOT,END}}$$

$$V_{\text{TOT,BEGIN}} = V_{\text{LB},i} + V_{\text{AQ},i}$$

$$V_{\text{LB},i} = \ell_{\text{LB},i} \cdot w_{\text{LB},i} \cdot h_{\text{LB},i} \qquad V_{\text{AQ},i} = \ell_{\text{AQ},i} \cdot w_{\text{AQ},i} \cdot h_{\text{AQ},i}$$

$$V_{\text{TOT,END}} = V_{\text{LB},f} + V_{\text{AQ},f}$$

$$V_{\text{LB},f} = \ell_{\text{LB},f} \cdot w_{\text{LB},f} \cdot h_{\text{LB},f} \qquad V_{\text{AQ},f} = \ell_{\text{AQ},f} \cdot w_{\text{AQ},f} \cdot h_{\text{AQ},f}$$

Read **Manager** relationships horizontally. Read **Associated Personnel** relationships vertically. When investigating a value, consider the possibly multiple relationships that constrain the value, just like when you consider the multiple clues that a single letter must sometimes satisfy in a crossword puzzle.

Example 5.1: Narrate an explanation for Example 4.1.

	Style	Skip statement	Tool	Equivalent parts (of statement & tool)	Populated tool / Point(s) (usually just give point(s) in written explanations)
1.	Pre-write	When asked to write an explanation, you often need not rewrite the problem statement or current line of algebraic work, which is typically already nearby.	Draw a **MAP** of all relevant relationships.	Fill in given values in your **MAP**. If given partial knowledge (e.g., you're not given a numerical value for ℓ, but you're given that ℓ increases), note the partial knowledge (e.g., draw an up arrow ↑ near ℓ).	Use given information you've just filled in to draw conclusions about quantities in your **MAP**.
2.	Default: Medium-long		The volume of water initially in the lunchbox (water's initial product of length, width, and height) converts fully into the volume of the water finally in the aquarium (water's final product of length, width, and height).	The aquarium's greater length and width mean that, at the end, the water at the bottom of the aquarium has greater length and width, respectively, than the length and width of the water initially in the lunchbox.	So, the water has less height in the aquarium.
3.	How to lengthen the medium-long version (typically not advised)		State each equation and name each quantity in Example 4.1's MAP.	You'd then need to explicitly enumerate each quantity in the MAP that equals 0.	
4.	Short			The water has a greater length and width when at the bottom of the aquarium than when filling the lunchbox.	So, the water has less height in the aquarium.

Example 6.1: What is $\sqrt{x^2}$? Alice proposes that $\sqrt{x^2} = x$, and Beth propose that $\sqrt{x^2} = |x|$.

	Shiftable input(s)	**Theoretical Expectations**		**Actual results and Measurements**
	x	By Alice's model $\sqrt{x^2} = x$,	By Beth's model $\sqrt{x^2} = |x|$,	$\sqrt{x^2}$
1.	-2	-2 😭	2	$\sqrt{(-2)^2} = \sqrt{4} = 2$
2.	-1	-1 😭	1	$\sqrt{(-1)^2} = \sqrt{1} = 1$
3.	0	0	0	$\sqrt{(0)^2} = \sqrt{0} = 0$
4.	1	1	1	$\sqrt{(1)^2} = \sqrt{1} = 1$
5.	2	2	2	$\sqrt{(2)^2} = \sqrt{4} = 2$

In the STEAM table, some of Alice's predictions are **inconsistent** with actual values (when x is negative), so **reject** Alice's model, $\sqrt{x^2} = x$.
In the STEAM table, all of Beth's predictions are **consistent** with actual values, so Beth's model $\sqrt{x^2} = |x|$ **remains in play**.

The STEAM table alone doesn't allow Beth's model to be "ruled in" since the STEAM table doesn't preclude the possibility of additional values of the shiftable input x resulting in values of $\sqrt{x^2}$ that turn out to fail to equal $|x|$.

Example 7.1: Condense $\ln(x) + 2\ln(y)$.

			Forbidden (excluded) values	
			Unstated (implied) exclusions	Listed (explicit) exclusions
1.	**Problem** expression, function, or relationship	$\ln(x) + 2\ln(y)$	The argument of a log must be positive, so $x > 0$ and $y > 0$.	(None given by problem statement)
2.	**Algebra STEPs** (possibly showing only **Statements**)	$\ln(x) + \ln(y^2)$	(Even in a sadistic high-school course, you can usually skip this section).	
3.	**Initial Naïve** answer	$\ln(xy^2)$	The argument of a log must be positive, so $xy^2 > 0$. x and y must simultaneously be non-zero (or else $xy^2 = 0$). Requiring $y \neq 0$ guarantees that $y^2 > 0$. So, the only remaining requirement to guarantee $xy^2 > 0$ is the requirement that $x > 0$. Summary: $x > 0$, $y \neq 0$.	Do the exclusions implied by the naïve answer exclude at least those values that were implicitly excluded by and explicitly excluded for the problem expression, function, or equation? The naïve answer's implied exclusion $x > 0$ correctly reproduces the problem's implied requirement that $x > 0$. No additional statement excluding x-values is needed. However, the naïve answer's implied exclusion $y \neq 0$ allows for the possibility that $y < 0$, which was excluded by the problem's implied requirement that $y > 0$. So, add a statement that $y > 0$.
			If the initial naïve answer is a candidate solution set, plug the candidate solutions back into the original problem relationship. Reject candidate statements that cause the problem relationship to be false.	

	Student response	Score	Expression	Exclusion statement
1.	$\ln(x) + 2\ln(y) = \ln(xy^2), y > 0$	**Full** credit ✓	**Correct**	**Correct** and **efficient**
2.	$\ln(x) + 2\ln(y) = \ln(xy^2), x > 0, y > 0$	Substantial **partial** credit ✗	**Correct**	**Correct**, but **inefficient** (stating $x > 0$ is unnecessary)
3.	$\ln(x) + 2\ln(y) = \ln(xy^2)$	Less **partial** credit ✗	**Correct**	**Incorrect**

For schools that enjoy playful torture, include the phrase "assume all variables are positive" in all problem statements for homework and class examples and never explicitly discuss the **PAINFUL** table above in class. Then, omit the phrase "assume all variables are positive" from exams.

EXERCISES

Lesson 1: STEP

Exercise 1.1: Solve for x in each of the following equations.
(a) $-5x + 2 = 12$
(b) $3(x - 1) = 15$
(c) $2x + 5x = 49$
(d) $4x + 5 = 9x$
(e) $x^2 = 9$
(f) $x^4 + 4 = 5x^2$

Lesson 2: STEP

Exercise 2.1: Narrate each step in the STEP tables from Exercise 1.1.

Lesson 3: SCAN

Exercise 3.1: Translate: "An object that travels for a duration of time t at constant speed v travels a distance d equal to the product of the duration and speed."

Exercise 3.2: Translate: "The initial value of a quantity plus the change in the value of that quantity equals the quantity's final value. The ratio of the change in price of a basket of goods through a year to the price of that basket of goods at the start of the year is the annual inflation rate measured using that basket."

Lesson 4: MAP

Exercise 4.1: A three-pound bag of trail mix contains 30% raisins by weight. A five-pound bag of trail mix contains 20% raisins by weight. Of the combined trail mix from the two bags, what percent by weight is raisins?

Exercise 4.2: The constant speed with which a boat travels from City A to City B is $5\,\frac{\text{miles}}{\text{hour}}$ less than the constant speed with which the boat travels from City B to City A. The length of the water route between Cities A and B is 50 miles, and the round-trip travel time (not including time spent docked) is 6 hours. What is the boat's speed during each leg of its trip?

Lesson 5: STEP

Exercise 5.1: Narrate your reasoning for exercise 4.1.

Exercise 5.2: Narrate your reasoning for exercise 4.2.

Lesson 6: STEAM

Exercise 6.1: Test the proposal that $\frac{x}{x} = 1$ and another proposed formula of your choosing for $\frac{x}{x}$.

Exercise 6.2: Test the proposal that $\left(\sqrt{x}\right)^2 = x$ and another proposed formula of your choosing for $\left(\sqrt{x}\right)^2$.

Lesson 7: PAINFUL

Exercise 7.1: Simplify the expression.
(a) $\frac{x^2}{x}$
(b) $\left(\sqrt{2x-1}\right)^2$

Exercise 7.2: Solve for x.

$$\log_{10}(x^2 - 6) = \log_{10}(x)$$

Exercise 7.3: Perform the indicated function operation.

(a) $f(x) = \sqrt{x-2} + 3$
$g(x) = x - \sqrt{x-2} - 3$
Find $f(x) + g(x)$

(b) $f(x) = \frac{x^2-6x-9}{x+2}$
$g(x) = \frac{x-3}{x^2-4}$
Find $\frac{f(x)}{g(x)}$