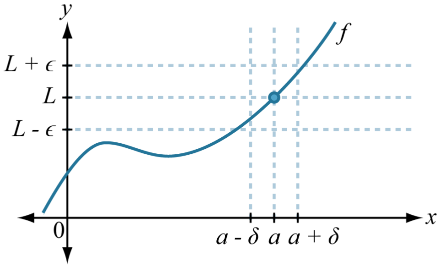
Calculating limits

$\epsilon - \delta$ *proofs* of existence and values of limits

Show that requiring $0 < |x - a| < \delta$ for some $\delta > 0$ ensures that $|f(x) - L| < \epsilon$ for desired $\epsilon > 0$, no matter how small ϵ is.

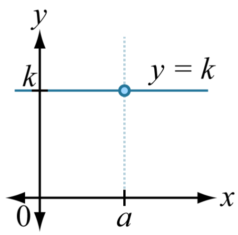
Tedious!

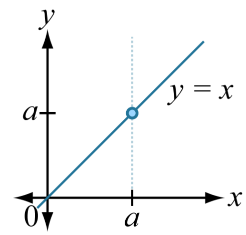


Limit identities (valid provided that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist and k is a constant)

$$\lim_{x \to a} k = k$$

$$\lim_{x \to a} x = a$$





$$\lim_{x \to a}[f(x) + g(x)] = \left[\lim_{x \to a} f(x)\right] + \left[\lim_{x \to a} g(x)\right]$$

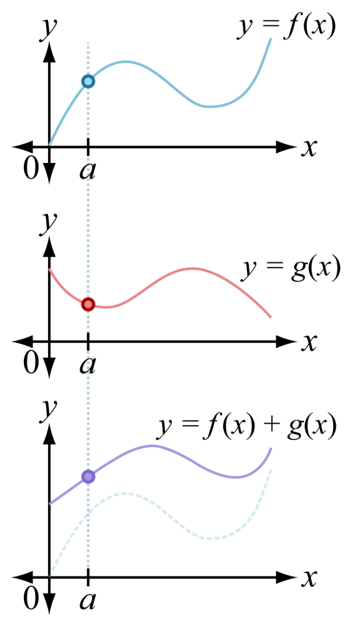
$$\lim_{x \to a}[f(x) \cdot g(x)] = \left[\lim_{x \to a} f(x)\right] \cdot \left[\lim_{x \to a} g(x)\right]$$

$$\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$$

$$\lim_{x \to a}[kf(x)] = k\left[\lim_{x \to a} f(x)\right]$$

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\left[\lim_{x \to a} f(x)\right]}{\left[\lim_{x \to a} g(x)\right]}, \qquad \lim_{x \to a} g(x) \neq 0$$

$$\lim_{x \to a} f\big(g(x)\big) = f\left(\lim_{x \to a} g(x)\right)$$



Calculating limits

Arithmetic *evaluation* of limits

1. Always consider **direct substitution** first, and frequently reconsider direct substitution after other strategies have been applied.

   If the function f is continuous (soon to be defined) at $x = a$, then $\lim_{x \to a} f(x) = f(a)$.

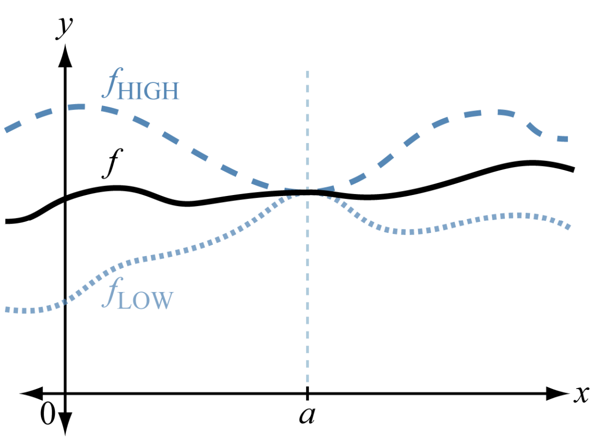
   We will learn that **polynomials, rational, and trigonometric functions** are continuous in their domains, so **"trying direct substitution" for such functions** reduces to **trying to find a value for** $f(a)$. If such a value can be found, then this value is the value of $\lim_{x \to a} f(x)$.

2. When considering direct substitution fails to provide a value for a limit, it might be helpful to consider the limit from the left and the limit from the right, $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$, respectively. Only when $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ does $\lim_{x \to a} f(x)$ exist.

3. When the process of considering direct substitution fails to provide a value for a limit, it might help to **apply a fancy algebraic manipulation**.
   a. Factor and cancel
   b. Multiply by radical conjugate
   c. Rationalize denominator
   d. Common denominator

Using definitions to recognize limits as other familiar objects

recognize a limit as a derivative (will make sense soon)



squeeze ("sandwich") theorem

Hypotheses

1. $f_{\text{LOW}}(x) \le f(x) \le f_{\text{HIGH}}(x)$ on $[b, a) \cup (a, c]$ where $b < a < c$.
2. $\lim_{x \to a} f_{\text{LOW}}(x)$ and $\lim_{x \to a} f_{\text{HIGH}}(x)$ exist and are equal.

Conclusion

$$\lim_{x \to a} f_{\text{LOW}}(x) = \lim_{x \to a} f(x) = \lim_{x \to a} f_{\text{HIGH}}(x)$$

Using geometric definitions of functions

Draw a figure that shows labeled geometric quantities including those quantities that are related by the function of interest. Study how some quantities change when a particular quantity (independent variable) is adjusted.

Example: Trigonometric limits

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad \lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$$