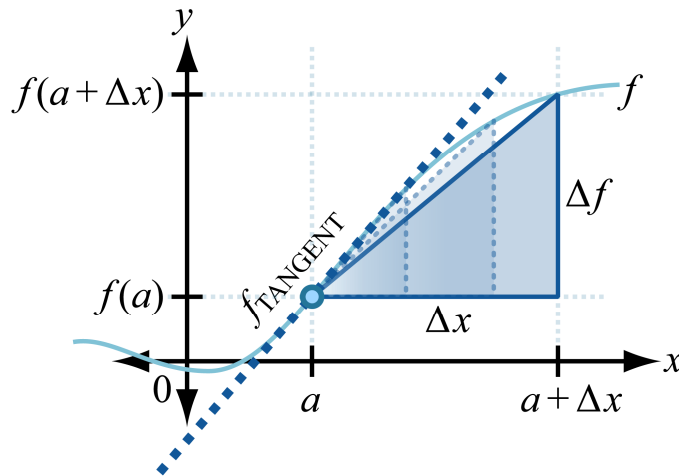


# Derivatives

The derivative is the slope of the tangent line



$$\text{Average slope} = \frac{\Delta f}{\Delta x}$$

$$\text{Instantaneous slope} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f}{\Delta x}$$

**Defn. 1** (lots of distributive multiplication):

$$f'(a) = \left. \frac{df}{dx} \right|_{x=a} := \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

**Defn. 2** (lots of factoring):

$$f'(a) = \left. \frac{df}{dx} \right|_{x=a} := \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$f_{\text{TANGENT}@a}(x) = f(a) + f'(a)(x - a)$$

## Notating a derivative of a function

### Review of functions

$f$	a name of Platonic ideal of a particular function
$x$	an arbitrary yet specific and particular value on the real number line
$f(x)$	the particular value that is associated with $x$ by the function $f$ ; can be expressed as an explicit association rule computing formula in terms of $x$

### Derivatives

$f'$ a.k.a. $\frac{df}{dx}$	a name of Platonic ideal of derivative of function named $f$
$f'(x)$ a.k.a. $\frac{df}{dx}(x)$	the particular value that is associated with $x$ by the function $f'$ ; can be expressed as an explicit association rule computing formula in terms of $x$

### Abuse of notation

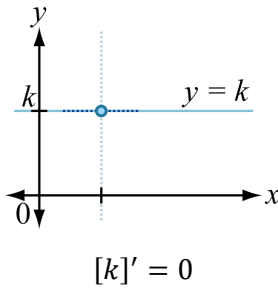
$[\text{expression in terms of } x]'^{\text{REPR}}$	where you see this, you can replace it with the association rule formula (in terms of $x$ ) corresponding to the derivative of the function whose association rule is named or written within the brackets to which $'^{\text{REPR}}$ is applied.
$\frac{d}{dx_{\text{REPR}}} [\text{expression in terms of } x]$	where you see this, you can replace it with the association rule formula (in terms of $x$ ) corresponding to the derivative of the function whose association rule is named or written within the brackets to which $\frac{d}{dx_{\text{REPR}}}$ is applied.

The symbols  $'^{\text{REPR}}$  and  $\frac{d}{dx_{\text{REPR}}}$  are distinct from the symbols  $'$  and  $\frac{d}{dx}$ . Abuse notation by omitting the distinguishing mark REPR.

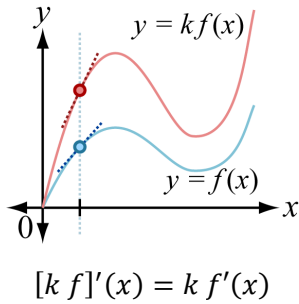
# Derivatives

Differentiation rules (valid provided that  $f'(x)$  and  $g'(x)$  exist)

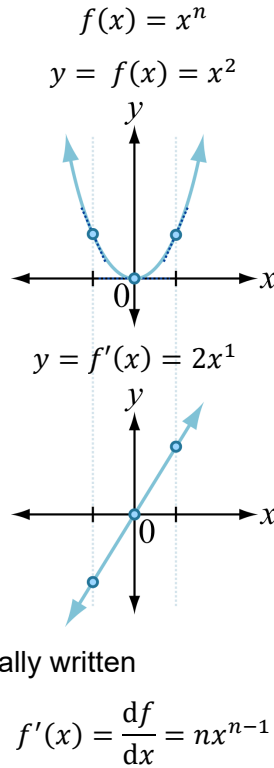
## Derivative of a constant



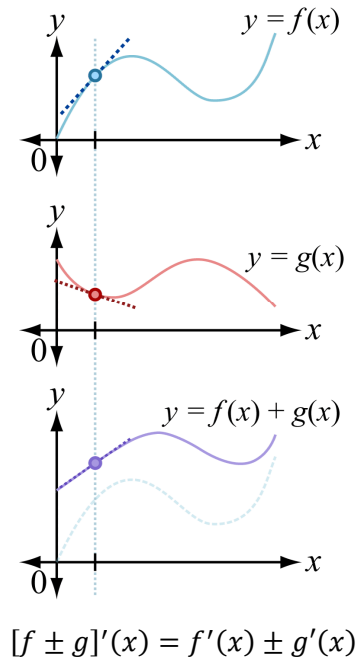
## Derivative of a constant times a function



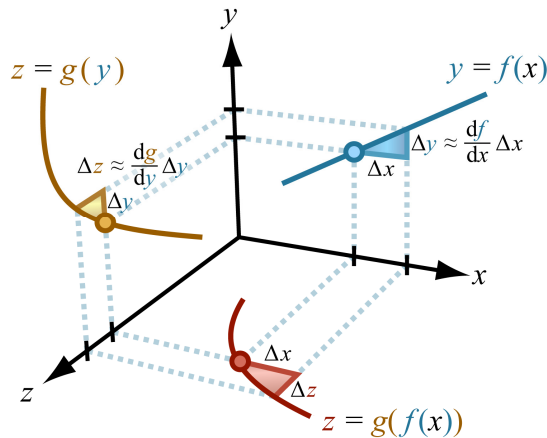
## Power rule



## Sum (or difference) rule



## Chain rule for derivative of a composite function



$$\frac{dg(f(x))}{dx} = \frac{dg(f)}{df} \cdot \frac{df(x)}{dx}$$

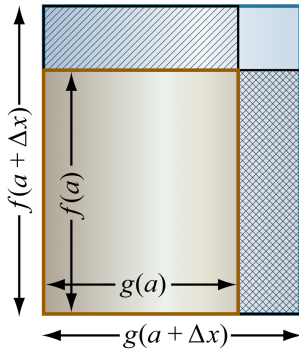
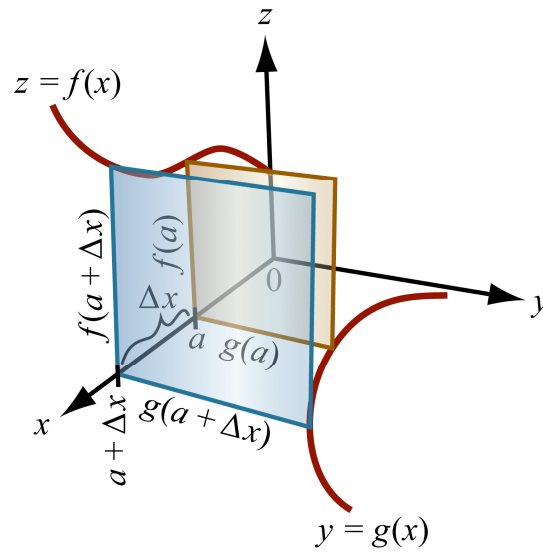
Shorthand:

$$\frac{dg}{dx} = \frac{dg}{df} \cdot \frac{df}{dx}$$

1. Use order of operations to identify sequence from innermost to outermost layer. Ask, "What do I do first to  $x$ ? Then what do I do next?" Continue until you have enumerated all operations. Distinguish the different layers using different colors (or, if colors are forbidden, using different bubble textures).
2. Write stuff'(x) = space to write a product.
3. Identify the currently outermost yet-to-be-differentiated layer and regard everything else in deeper layers as " $x$  for now".
4. Differentiate the currently outermost yet-to-be-differentiated layer with respect to " $x$  for now".
5. Write the expression obtained in step 4 as a factor in the space you set aside in step 2.
6. What was the " $x$  for now" in step 4? Was it  $x$  itself? If not, repeat steps 3-6. If yes, stop.

# Derivatives

## Product rule



$$\begin{aligned} \frac{d[f(x)g(x)]}{dx} \Big|_{x=a} &= \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x)g(a + \Delta x) - f(a)g(a)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\text{Area(Beam)} + \text{Area(Pillar)}}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\Delta f \cdot g(a) + f(a) \cdot \Delta g}{\Delta x} \\ &= \frac{df}{dx} \Big|_{x=a} g(a) + f(a) \frac{dg}{dx} \Big|_{x=a} \end{aligned}$$

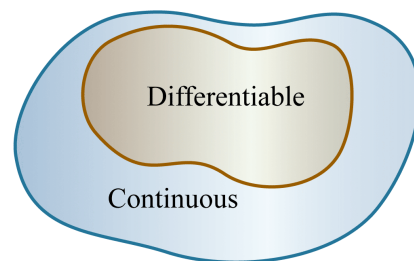
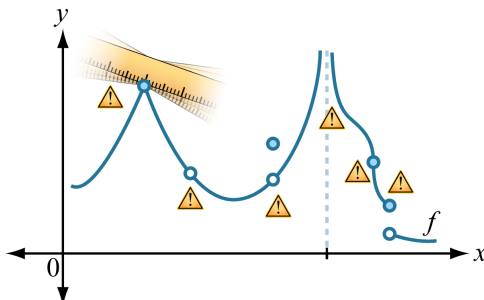
$$[fg]'(x) = f'(x)g(x) + f(x)g'(x)$$

## Quotient rule

$$\left[ \frac{f}{g} \right]'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

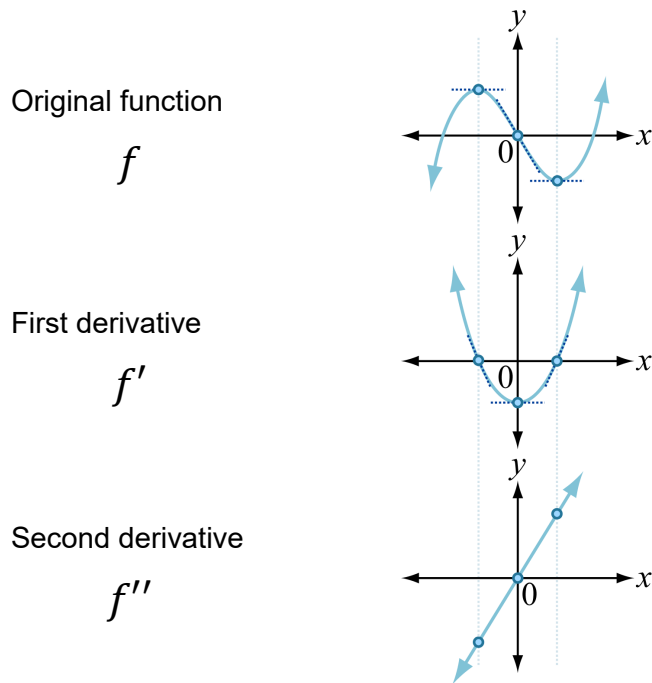
## Derivatives do not always exist

Saying that  $f$  is differentiable at  $x = a$  means that  $f'(a)$  exists. Saying that  $f$  is differentiable on an open interval means that  $f$  is differentiable at every  $x$ -value in the interval.



# Derivatives

## Higher-order derivatives



### Notation

Lagrange	Leibniz
$f$	$f$
$f'$	$\frac{df}{dx}$
$f''$	$\frac{d^2f}{dx^2}$
$f'''$	$\frac{d^3f}{dx^3}$
$f^{(4)}$	$\frac{d^4f}{dx^4}$
$\vdots$	$\vdots$
$f^{(n)}$	$\frac{d^n f}{dx^n}$

### Explicit differentiation

$$\begin{aligned}
 xy &= 3 \\
 y &= \frac{3}{x} \\
 y' &= 3 \left( -\frac{1}{x^2} \right) = -\frac{3}{x^2} \\
 &= -\frac{(xy)}{x^2} \\
 &= -\frac{y}{x}
 \end{aligned}$$

vs.

### Implicit differentiation

$$\begin{aligned}
 x \cdot y(x) &= 3 \\
 [x \cdot y(x)]' &= [3]' \\
 1 \cdot y(x) + x \cdot y'(x) &= 0 \\
 x \cdot y'(x) &= -y(x) \\
 y'(x) &= -\frac{y(x)}{x}
 \end{aligned}$$