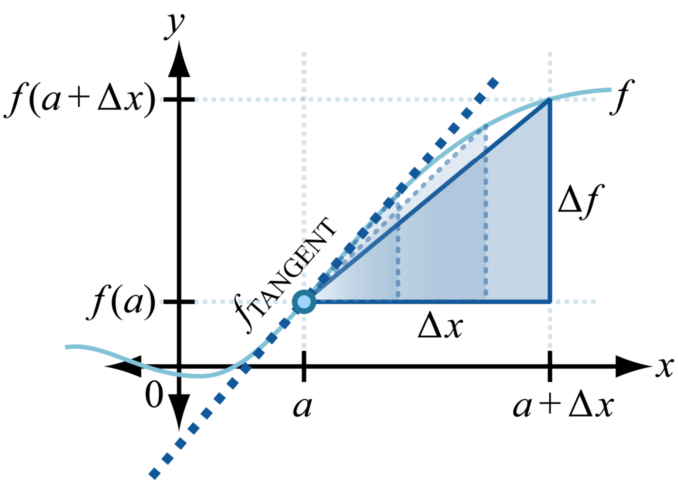
Derivatives

The derivative is the slope of the tangent line



$$\text{Average slope} = \frac{\Delta f}{\Delta x}$$

$$\text{Instantaneous slope} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$$

Defn. 1 (lots of distributive multiplication):

$$f'(a) = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a} := \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

Defn. 2 (lots of factoring):

$$f'(a) = \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a} := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$f_{\text{TANGENT @}a}(x) = f(a) + f'(a)(x - a)$$

Notating a derivative of a function

Review of functions

f	a name of Platonic ideal of a particular function
x	an arbitrary yet specific and particular value on the real number line
$f(x)$	the particular value that is associated with x by the function f; can be expressed as an explicit association rule computing formula in terms of x

Derivatives

f' a.k.a. $\frac{\mathrm{d}f}{\mathrm{d}x}$	a name of Platonic ideal of derivative of function named f
$f'(x)$ a.k.a. $\frac{\mathrm{d}f}{\mathrm{d}x}(x)$	the particular value that is associated with x by the function f'; can be expressed as an explicit association rule computing formula in terms of x

Abuse of notation

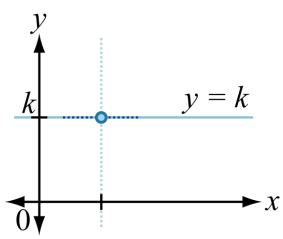
$[\text{expression in terms of } x]'^{\text{REPR}}$	where you see this, you can replace it with the association rule formula (in terms of x) corresponding to the derivative of the function whose association rule is named or written within the brackets to which $'^{\text{REPR}}$ is applied.
$\frac{\mathrm{d}}{\mathrm{d}x_{\text{REPR}}}[\text{expression in terms of } x]$	where you see this, you can replace it with the association rule formula (in terms of x) corresponding to the derivative of the function whose association rule is named or written within the brackets to which $\frac{\mathrm{d}}{\mathrm{d}x_{\text{REPR}}}$ is applied.

The symbols $'^{\text{REPR}}$ and $\frac{\mathrm{d}}{\mathrm{d}x_{\text{REPR}}}$ are distinct from the symbols $'$ and $\frac{\mathrm{d}}{\mathrm{d}x}$. Abuse notation by omitting the distinguishing mark REPR.

Derivatives

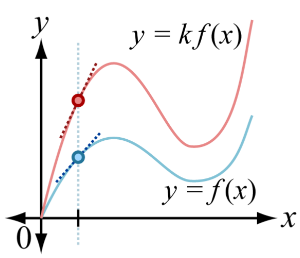
Differentiation rules (valid provided that $f'(x)$ and $g'(x)$ exist)

Derivative of a constant



$$[k]' = 0$$

Derivative of a constant times a function

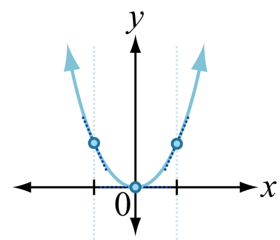


$$[k\,f]'(x) = k\,f'(x)$$

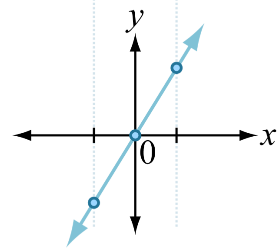
Power rule

$$f(x) = x^n$$

$$y = f(x) = x^2$$



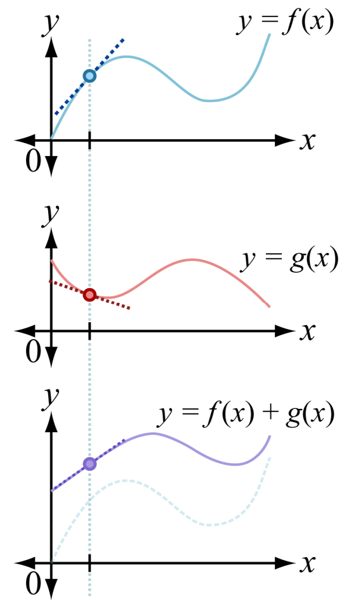
$$y = f'(x) = 2x^1$$



Usually written

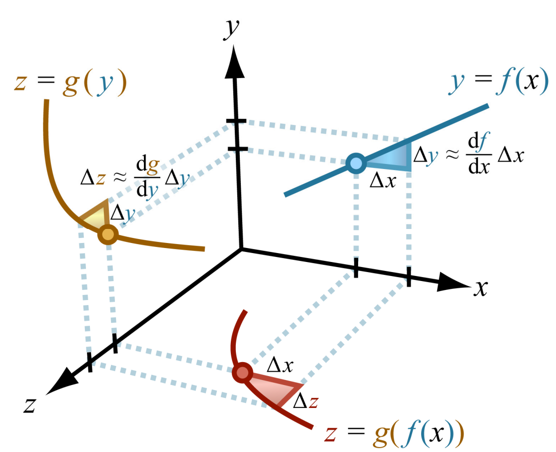
$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = nx^{n-1}$$

Sum (or difference) **rule**



$$[f \pm g]'(x) = f'(x) \pm g'(x)$$

Chain rule for derivative of a composite function



$$\frac{\mathrm{d}g\big(f(x)\big)}{\mathrm{d}x} = \frac{\mathrm{d}g(f)}{\mathrm{d}f} \cdot \frac{\mathrm{d}f(x)}{\mathrm{d}x}$$

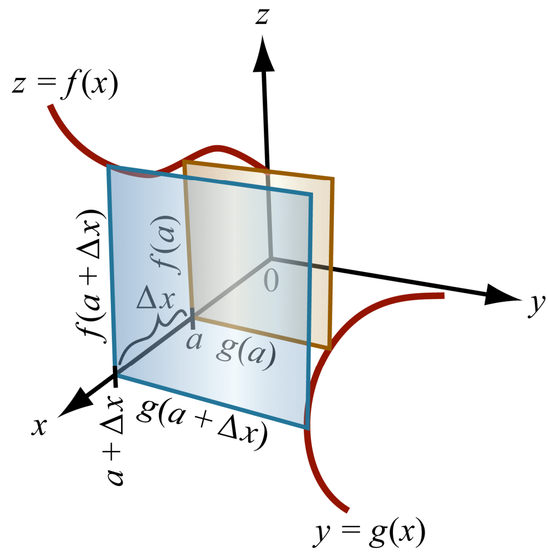
Shorthand:

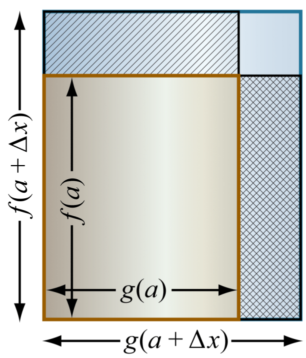
$$\frac{\mathrm{d}g}{\mathrm{d}x} = \frac{\mathrm{d}g}{\mathrm{d}f} \cdot \frac{\mathrm{d}f}{\mathrm{d}x}$$

1. Use order of operations to identify sequence from innermost to outermost layer. Ask, “What do I do first to x? Then what do I do next?” Continue until you have enumerated all operations. Distinguish the different layers using different colors.
2. Write $\text{stuff}'(x) = 1 \cdot \text{space for more writing.}$
3. Identify the currently outermost yet-to-be-differentiated layer and regard everything else in deeper layers as a single abstract variable.
4. Differentiate the currently outermost yet-to-be-differentiated layer with respect to everything else in deeper layers regarded as a single abstract variable.
5. Multiply the until-now formed product of factors by the expression obtained in step 4.
6. What was the single abstract variable with which you “with respected to” in step 4? Was it x itself? If not, repeat steps 3-6. If yes, stop.

Derivatives

Product rule





$$\left.\frac{\mathrm{d}[f(x)g(x)]}{\mathrm{d}x}\right|_{x=a} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x)g(a+\Delta x) - f(a)g(a)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\mathrm{Area(Beam)} + \mathrm{Area(Pillar)}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta f \cdot g(a) + f(a) \cdot \Delta g}{\Delta x}$$

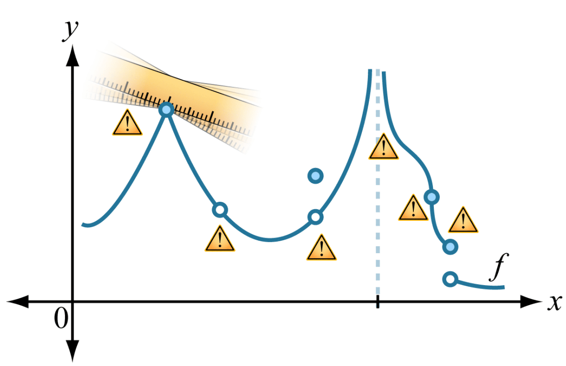
$$= \left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a} g(a) + f(a) \left.\frac{\mathrm{d}g}{\mathrm{d}x}\right|_{x=a}$$

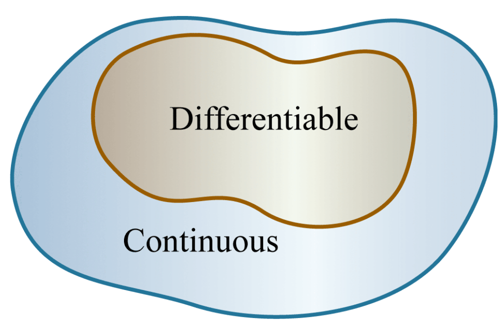
$$[fg]'(x) = f'(x)g(x) + f(x)g'(x)$$

Quotient rule

$$\left[\frac{f}{g}\right]'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}, \quad g(x) \neq 0$$

Derivatives do not always exist





Derivatives

Higher-order derivatives

Original function

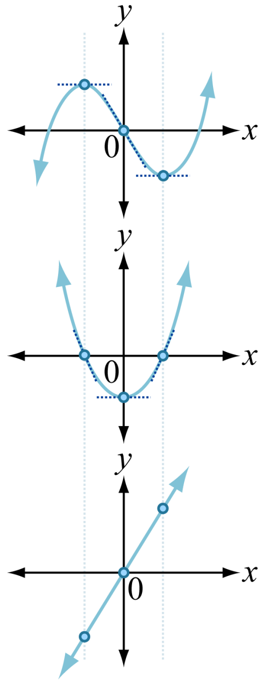
f

First derivative

f'

Second derivative

f''



Notation

Lagrange	Leibniz
f	f
f'	$\frac{\mathrm{d}f}{\mathrm{d}x}$
f''	$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$
f'''	$\frac{\mathrm{d}^3 f}{\mathrm{d}x^3}$
$f^{(4)}$	$\frac{\mathrm{d}^4 f}{\mathrm{d}x^4}$
$\vdots$	$\vdots$
$f^{(n)}$	$\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$

Explicit differentiation vs. **Implicit differentiation**

Explicit differentiation:

$$xy = 3$$

$$y = \frac{3}{x}$$

$$y' = 3\left(-\frac{1}{x^2}\right) = -\frac{3}{x^2}$$

$$= -\frac{(xy)}{x^2}$$

$$= -\frac{y}{x}$$

Implicit differentiation:

$$x \cdot y(x) = 3$$

$$[x \cdot y(x)]' = [3]'$$

$$1 \cdot y(x) + x \cdot y'(x) = 0$$

$$x \cdot y'(x) = -y(x)$$

$$y'(x) = -\frac{y(x)}{x}$$