l'Hôpital's rule

Theory

Motivating special case

Example: Each of functions f and g has the following properties:

- Twice differentiable
- Root at $x = a$
- Equal to local linearization at $x = a$

$$f(x) = \overbrace{f(a)}^{0} + f'(a)(x-a)$$

$$g(x) = \overbrace{g(a)}^{0} + g'(a)(x-a)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)(x-a)}{g'(a)(x-a)} \overset{\text{D.S.}}{\triangleq} \frac{0}{0}$$

$$= \lim_{x\to a}\frac{f'(a)}{g'(a)} = \frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}$$

Steps

1. Write the given limit

$$\lim_{x\to a}\frac{f(x)}{g(x)}$$

2. Determine whether direct substitution (or "direct substitution" for limits "at" $\pm\infty$) yields a value for the limit.
   a. If yes (e.g. when taking limit of continuous function at a value of x), stop.
   b. If, instead, one of the indeterminate forms ${}^{0}/_{0}$, $\pm{}^{\infty}/_{\pm\infty}$, or $\pm{}^{\infty}/_{\mp\infty}$ results, go on to step 3.

3. Try to find a value for

$$\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

4. If the limit in step 3 has a value (or is $\pm\infty$), then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Clever application

Product to quotient

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{{}^{1}/_{x}} \overset{\text{D.S.}}{\triangleq} \frac{-\infty}{+\infty}$$

$$\lim_{x\to 0^+}\frac{\ln x}{{}^{1}/_{x}} \overset{\text{l'H}}{\triangleq} \lim_{x\to 0^+}\frac{(\ln x)'}{({}^{1}/_{x})'} = \lim_{x\to 0^+}\frac{{}^{1}/_{x}}{-{}^{1}/_{x^2}} = \lim_{x\to 0^+} -x \overset{\text{D.S.}}{\triangleq} 0$$

Therefore,

$$\lim_{x\to 0^+} x\ln x = 0$$

Power to quotient

$$\lim_{x\to\infty} x^{\frac{1}{x}} \overset{\text{"D.S."}}{\triangleq} \infty^0$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty}\exp\left(\ln x^{\frac{1}{x}}\right) \overset{\text{COMP}}{\triangleq} \exp\left(\lim_{x\to\infty}\ln x^{\frac{1}{x}}\right)$$

$$\lim_{x\to\infty}\ln x^{\frac{1}{x}} = \lim_{x\to\infty}\frac{1}{x}\ln x = \lim_{x\to\infty}\frac{\ln x}{x} \overset{\text{"D.S."}}{\triangleq} \frac{+\infty}{+\infty}$$

$$\lim_{x\to\infty}\frac{\ln x}{x} \overset{\text{l'H}}{\triangleq} \lim_{x\to\infty}\frac{(\ln x)'}{(x)'} = \lim_{x\to\infty}\frac{{}^{1}/_{x}}{1} \overset{\text{"D.S."}}{\triangleq} 0$$

Therefore,

$$\lim_{x\to\infty}\ln x^{\frac{1}{x}} = 0$$

Therefore,

$$\exp\left(\lim_{x\to\infty}\ln x^{\frac{1}{x}}\right) = \exp(0) = 1$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = 1$$