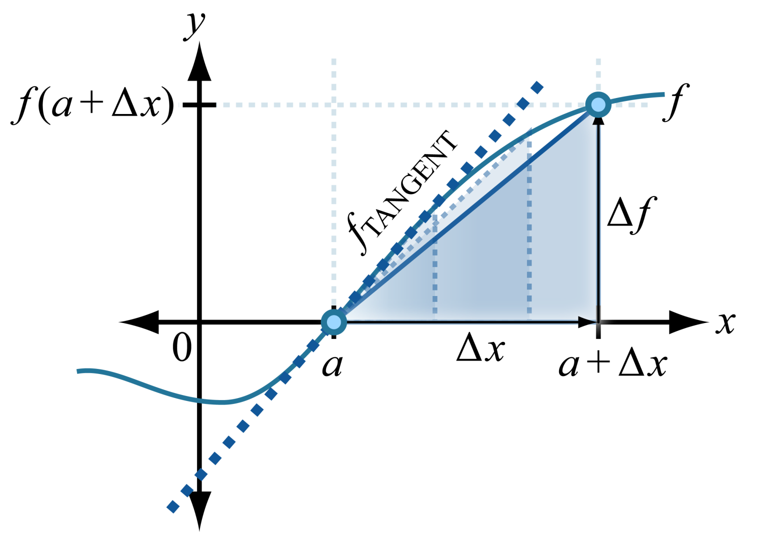
l'Hôpital's rule

Artificial motivating example

Each of functions f and g has the following properties:

- Root at $x = a$
- Equal to linearization at $x = a$



Even though f (above) differs from its $x = a$ linearization, the graph of f resembles, near $x = a$, that linearization.

$$f(x) = \overbrace{f(a)}^{0} + f'(a)(x-a) \qquad\qquad f'(x) = f'(a)$$

so

$$g(x) = \overbrace{g(a)}^{0} + g'(a)(x-a) \qquad\qquad g'(x) = g'(a)$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)(x-a)}{g'(a)(x-a)} \overset{\substack{\text{Attempting}\\ \text{quotient rule}\\ \text{for limits}}}{\cong} \frac{\overbrace{\lim_{x\to a} f'(a)(x-a)}^{0}}{\underbrace{\lim_{x\to a} g'(a)(x-a)}_{0}} = \frac{0}{0}$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} \overset{\substack{\text{Try factor}\\ \text{and cancel}}}{\cong} \lim_{x\to a}\frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}} = \lim_{x\to a}\frac{f'(a)}{g'(a)} = \frac{f'(a)}{g'(a)}$$

Notice that another limit yields the same result

$$\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f'(a)}{g'(a)} = \frac{f'(a)}{g'(a)}$$

So

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

l'Hopital's rule

Hypothesis

1. f and g are differentiable
2. $g'(x) \neq 0$ near a (except possibly at a)
3. Either of the following:
   a. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$
   b. $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$
4. $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ has a value (or is $\pm\infty$)

Conclusion

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Steps

1. Write the given limit
$$\lim_{x\to a}\frac{f(x)}{g(x)}$$
2. Determine whether direct substitution (or "direct substitution" for limits "at" $\pm\infty$) yields a value (or $\pm\infty$) for the limit.
   a. If yes (e.g. when taking limit of continuous function at a value of x), stop.
   b. If, instead, one of the indeterminate forms ${}^{0}/_{0}$, ${}^{\pm\infty}/_{\pm\infty}$, or ${}^{\pm\infty}/_{\mp\infty}$ results, go on to step 3.
3. Report the limits leading to the indeterminate form (don't usually need to mention differentiability):
$$\begin{cases}\lim_{x\to a} f(x) = 0 \text{ or } \pm\infty\\ \lim_{x\to a} g(x) = 0 \text{ or } \pm\infty\end{cases}$$
4. Provisionally try to find a value for
$$\lim_{x\to a}\frac{f'(x)}{g'(x)}$$
5. If the limit in step 4 has a value (or is $\pm\infty$), write a justifying note above and the rest of the equation to the left of your work from step 4. You can also annotate the equal sign (not seen in AP Calculus scoring guidelines).

   Using l'Hôpital's rule,
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \cdots$$

Product to quotient

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{{}^{1}/_{x}} \overset{\text{D.S.}}{\cong} \frac{-\infty}{+\infty}$$

$$\begin{cases}\lim_{x\to 0^+}\ln x = -\infty\\ \lim_{x\to 0^+}{}^{1}/_{x} = +\infty\end{cases}$$

Using l'Hôpital's rule,

$$\lim_{x\to 0^+}\frac{\ln x}{{}^{1}/_{x}} = \lim_{x\to 0^+}\frac{(\ln x)'}{({}^{1}/_{x})'} = \lim_{x\to 0^+}\frac{{}^{1}/_{x}}{-{}^{1}/_{x^2}} = \lim_{x\to 0^+} -x \overset{\text{D.S.}}{\cong} 0$$

Therefore,

$$\lim_{x\to 0^+} x\ln x = 0$$

Power to quotient

$$L = \lim_{x\to\infty} x^{\frac{1}{x}} \overset{\text{"D.S."}}{\cong} \infty^0$$

$$\ln L = \ln\left(\lim_{x\to\infty} x^{\frac{1}{x}}\right) = \lim_{x\to\infty}\ln x^{\frac{1}{x}} = \lim_{x\to\infty}\frac{1}{x}\ln x = \lim_{x\to\infty}\frac{\ln x}{x}$$

$$\begin{cases}\lim_{x\to\infty}\ln x = +\infty\\ \lim_{x\to\infty} x = +\infty\end{cases}$$

Using l'Hôpital's rule,

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{(\ln x)'}{(x)'} = \lim_{x\to\infty}\frac{{}^{1}/_{x}}{1} \overset{\text{"D.S."}}{\cong} 0$$

Therefore,

$$\ln L = 0$$

$$L = e^0 = 1$$

$$\lim_{x\to\infty} x^{\frac{1}{x}} = 1$$