Initial value problems & differential equations

Goal	To identify candidate(s) for **undetermined function** y represented by formula $y(x)$
Possible **demands** on $y(x)$	A **differential equation (DE)** $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ specifies that if the graph of y were to pass through the point (a, b), then the slope of the plotted curve at the point (a, b) would be $f(a, b)$.
	An **initial condition (IC)** is a specified point (x_i, y_i) through which the graph of y passes.

Demand(s) on $y(x)$	Set of $y(x)$ surviving logical elimination
Require $y(x)$ to **satisfy** given **DE**	**general solution** – represents family of functions
Require $y(x)$ to **satisfy** given **DE** Require $y(x)$ to **be satisfied by** given **IC**	**particular solution** – represents one function

Analytic approaches

Given: Differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = y' = f(x, y)$ (for the method outlines below, $f(x, y) = f(x)$)
Initial condition $y(x_i) = y_i$

Method I: Antidifferentiate, determine integration constant

$$\int y' \,\mathrm{d}x = \int f(x) \,\mathrm{d}x$$

$$y = F(x) + C$$

Substitute initial condition

$$y_i = F(x_i) + C$$

to determine C.

Method II: Calculate definite integrals with initial condition substituted into bounds

$$\int_{x=x_i}^{x=x_f} y' \,\mathrm{d}x = \int_{x=x_i}^{x=x_f} f(x) \,\mathrm{d}x$$

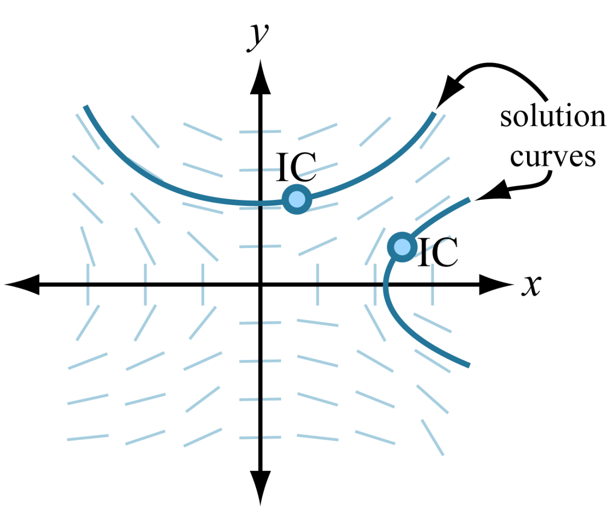
$$y(x_f) - y(x_i) = F(x_f) - F(x_i)$$

$$y(x_f) = y_i + F(x_f) - F(x_i)$$

Drop the subscript f:

$$y(x) = y_i + F(x) - F(x_i)$$

Graphical approach: Slope fields



At each of a set of representative points, draw a small **quiver whose slope is prescribed by the DE**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

From a point (use IC (x_i, y_i) if given), **trace out a path locally parallel to every slope segment it passes near/through**.

Tracing a solution curve through an IC is equivalent to using the DE and IC analytically to obtain $y(x)$.

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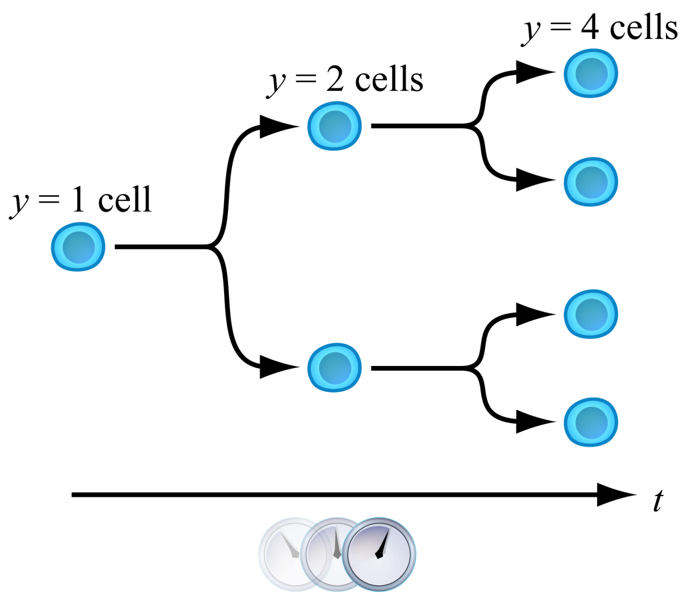
Population growth and decay



$$\frac{\mathrm{d}y}{\mathrm{d}x} = ky$$

Separation of variables

$$\int_{y=y_i}^{y=y_f} \frac{1}{y}\,\mathrm{d}y = k \int_{t=t_i}^{t=t_f} \mathrm{d}t$$

$$[\ln|y|]_{y=y_i}^{y=y_f} = k[t]_{t=t_i}^{t=t_f}$$

assuming that $y > 0$,

$$\ln(y_f) - \ln(y_i) = k[t_f - t_i]$$

$$\ln\left(\frac{y_f}{y_i}\right) = k(t_f - t_i)$$

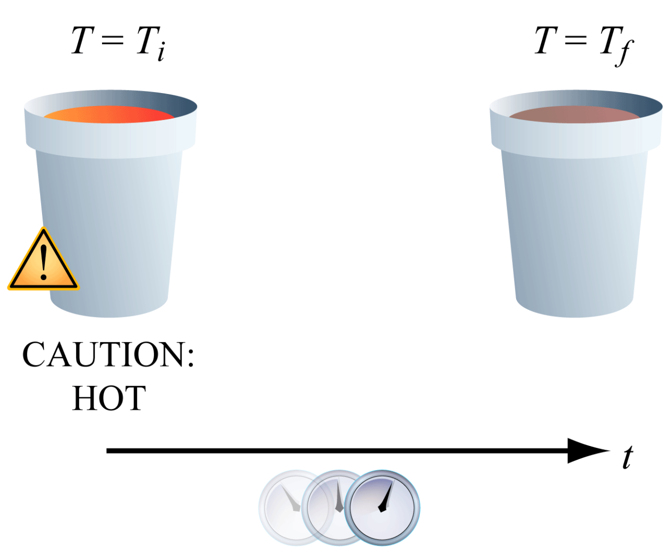
$$\frac{y_f}{y_i} = e^{k(t_f - t_i)}$$

$$y_f = y_i e^{k(t_f - t_i)}$$

Drop the subscript f:

$$y = y_i e^{k(t - t_i)}$$

Heating and cooling



$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T_{\mathrm{ROOM}} - T)$$

Separation of variables

$$\int_{T=T_i}^{T=T_f} \frac{1}{T_{\mathrm{ROOM}} - T}\,\mathrm{d}T = k \int_{t=t_i}^{t=t_f} \mathrm{d}t$$

$$[-\ln|T_{\mathrm{ROOM}} - T|]_{T=T_i}^{T=T_f} = k(t_f - t_i)$$

The absolute value bars can be removed and replaced with either $+(\quad)$ or $-(\quad)$, depending on whether $T_{\mathrm{ROOM}} - T$ is > 0 or < 0 for the situation being studied.