Integration by parts

Derivation from product rule for differentiation

$$\frac{\mathrm{d}(uv)}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}v + u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}(uv)}{\mathrm{d}x}$$

$$u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}(uv)}{\mathrm{d}x} - \frac{\mathrm{d}u}{\mathrm{d}x}v$$

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{\mathrm{d}(uv)}{\mathrm{d}x}\,\mathrm{d}x - \int \frac{\mathrm{d}u}{\mathrm{d}x}v\,\mathrm{d}x$$

$$\int u\,\mathrm{d}v = \int \mathrm{d}(uv) - \int v\,\mathrm{d}u$$

Formula

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

Heuristics for choosing u

Can you choose $u = \mathrm{stuff}$ in a way that satisfies some of the following criteria?

- u can be differentiated, preferably without a more complicated result
- $\mathrm{d}v$ can be integrated, preferably without a more complicated result

Type of function	Have differentiation formula?	Have antiderivative formula?*
Logarithms	Yes	No
Inverse trigonometric functions	Yes	No
Power law and **p**olynomial functions	Yes	Yes
Exponential functions	Yes	Yes
Trigonometric functions	Yes	Yes

* Before knowing about integration by parts

Integration by parts

Example: **Single application**

$$\int x\cos x\,\mathrm{d}x$$

$$u = x \qquad \mathrm{d}v = \cos x\,\mathrm{d}x$$
$$\mathrm{d}u = \mathrm{d}x \qquad v = \sin x$$

$$\int x\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x$$
$$= x\sin x + \int(-\sin x)\mathrm{d}x$$
$$= x\sin x + \cos x + C$$

Example: **Tabular method**

$$\int x^3e^x\,\mathrm{d}x$$

		Derivatives		**I**ntegrals
$+$	$\rightarrow$	$u = x^3$	$\searrow$	$\mathrm{d}v = e^x\,\mathrm{d}x$
$-$	$\rightarrow$	$3x^2$	$\searrow$	e^x
$+$	$\rightarrow$	$6x$	$\searrow$	e^x
$-$	$\rightarrow$	6	$\searrow$	e^x
$+$	$\rightarrow$	0	$\searrow$	e^x

Example: **Repeated application**

$$\int x^3e^x\,\mathrm{d}x$$

$$u = x^3 \qquad \mathrm{d}v = e^x\,\mathrm{d}x$$
$$\mathrm{d}u = 3x^2\mathrm{d}x \qquad v = e^x$$

$$\int x^3e^x\,\mathrm{d}x = x^3e^x - \int e^x\,3x^2\,\mathrm{d}x$$
$$= x^3e^x - 3\int x^2e^x\,\mathrm{d}x$$

$$\int x^2e^x\,\mathrm{d}x$$

$$u = x^2 \qquad \mathrm{d}v = e^x\,\mathrm{d}x$$
$$\mathrm{d}u = 2x\,\mathrm{d}x \qquad v = e^x$$

$$\int x^2e^x\,\mathrm{d}x = x^2e^x - \int e^x\,2x\,\mathrm{d}x$$
$$= x^2e^x - 2\int x\,e^x\,\mathrm{d}x$$

$$\int xe^x\,\mathrm{d}x$$

$$u = x \qquad \mathrm{d}v = e^x\,\mathrm{d}x$$
$$\mathrm{d}u = \mathrm{d}x \qquad v = e^x$$

$$\int xe^x\,\mathrm{d}x = xe^x - \int e^x\,\mathrm{d}x$$
$$= xe^x - e^x + C_1$$

$$\int x^3e^x\,\mathrm{d}x = x^3e^x - 3[x^2e^x - 2(xe^x - e^x + C_1)]$$
$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

Example: **Apparently endless** repetition

$$\int e^x\sin x\,\mathrm{d}x$$

$$u = e^x \qquad \mathrm{d}v = \sin x\,\mathrm{d}x$$
$$\mathrm{d}u = e^x\,\mathrm{d}x \qquad v = -\cos x$$

$$\int e^x\sin x\,\mathrm{d}x = e^x(-\cos x) - \int(-\cos x)\,e^x\,\mathrm{d}x$$
$$= -e^x\cos x + \int e^x\cos x\,\mathrm{d}x$$

$$\int e^x\cos x\,\mathrm{d}x$$

$$u = e^x \qquad \mathrm{d}v = \cos x\,\mathrm{d}x$$
$$\mathrm{d}u = e^x\,\mathrm{d}x \qquad v = \sin x$$

$$\int e^x\cos x\,\mathrm{d}x = e^x\sin x - \int \sin x\,e^x\,\mathrm{d}x$$
$$= e^x\sin x - \int e^x\sin x\,\mathrm{d}x$$

$$\int e^x\sin x\,\mathrm{d}x = -e^x\cos x + \left[e^x\sin x - \int e^x\sin x\,\mathrm{d}x\right]$$
$$= e^x(\sin x - \cos x) - \int e^x\sin x\,\mathrm{d}x$$

$$2\int e^x\sin x\,\mathrm{d}x = e^x(\sin x - \cos x) + C_1$$

$$\int e^x\sin x\,\mathrm{d}x = \frac{e^x(\sin x - \cos x)}{2} + C$$