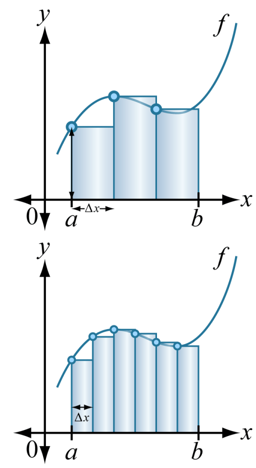
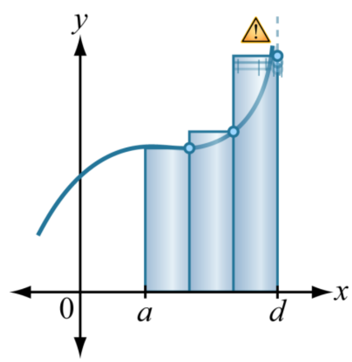
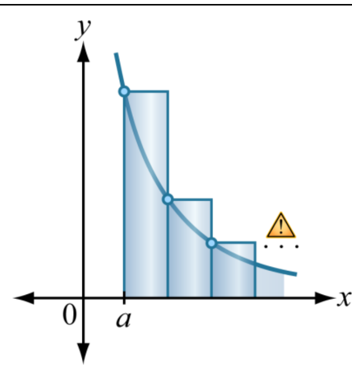
Improper integrals

Previously, we defined the definite integral in terms of a limit of a Riemann sum (sum over areas of rectangular subregions).

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i^*)\,\Delta x\,, \qquad N = \frac{b-a}{\Delta x}$$



The examples below illustrate some ways in which we can write down symbols using the integral sign that, when strictly following the definition of a definite integral in terms of the limit of a Riemann sum, fail to produce a number.

Example nonsense symbol	Cartoon	Issue
$\int_a^d \left(\begin{array}{c}\text{formula with}\\ \text{infinite discontinuity}\\ \text{at } x=d\end{array}\right) \mathrm{d}x$		Suppose we were to construct a Riemann sum using the RRAM. This would require constructing rectangular subregions, with the rightmost subregion having its right edge at $x = d$. However, the integrand has no finite height at $x = d$ because the integrand has here an **infinite discontinuity**. Thus, the rectangular subregion that would have had its right edge at $x = d$ cannot be constructed.
$\int_a^{+\infty} f(x)\,\mathrm{d}x$		**Unbounded interval** of integration prevents construction of a Riemann sum using a finite number of rectangular subregions of finite width

Both cartoons illustrate shaded regions that might have finite area (need only finite amount of paint to cover). This intuition conflicts with the fact that the notated definite integrals fail to yield numbers when interpreted directly using the definition of the definite integral in terms of a limit of a Riemann sum. To address this conflict, we declare new meanings for integral symbols like these so that we might be able to assign numbers to such symbols.

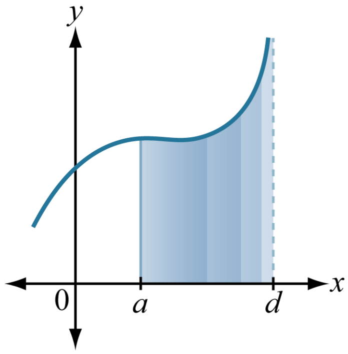
Improper integrals

Unbounded largeness of function (function has infinite discontinuity)

$$\int_a^b f(x)\,\mathrm{d}x$$

is improper when $f(x) \to \pm\infty$ as x approaches one or more values in $[a, b]$.

Vertical asymptote at one end

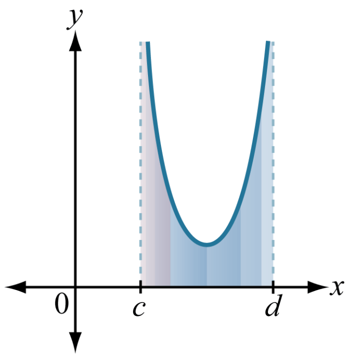


If $f(x)$ becomes unboundedly large as $x \to d^-$,

$$\int_a^d f(x)\,\mathrm{d}x := \lim_{b \to d^-} \int_a^b f(x)\,\mathrm{d}x$$

provided that the limit exists.

Vertical asymptotes at both ends

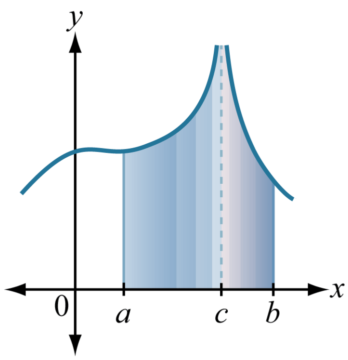


If $f(x)$ becomes unboundedly large as $x \to c^+$ and as $x \to d^-$,

$$\int_c^d f(x)\,\mathrm{d}x := \lim_{a \to c^+} \int_a^g f(x)\,\mathrm{d}x + \lim_{b \to d^-} \int_g^b f(x)\,\mathrm{d}x$$

provided that both limits exist for some g such that $c < g < d$.

Vertical asymptote in interior



If $f(x)$ becomes unboundedly large as $x \to c$, where $a < c < b$,

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{g \to c^-} \int_a^g f(x)\,\mathrm{d}x + \lim_{h \to c^+} \int_h^b f(x)\,\mathrm{d}x$$

provided that both limits exist.

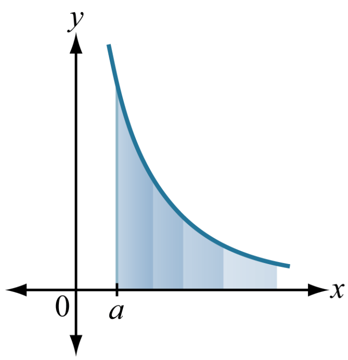
Improper integrals

Interval of integration cannot be bounded by a finite interval

$$\int_a^b f(x)\,\mathrm{d}x$$

is improper when a, b, or both a and b are $\pm\infty$ symbols.

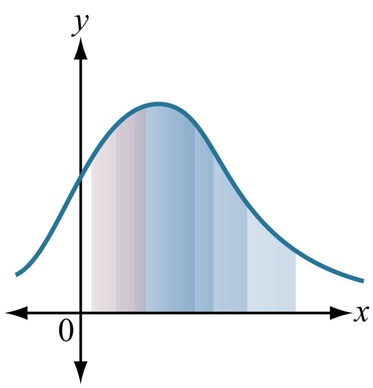
Precisely one boundary expression contains an infinity



$$\int_a^{+\infty} f(x)\,\mathrm{d}x := \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$$

provided that the limit exists.

Both boundary expressions contain infinities



$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x := \lim_{a\to-\infty}\int_a^c f(x)\,\mathrm{d}x + \lim_{b\to\infty}\int_c^b f(x)\,\mathrm{d}x$$

provided that both limits exist.