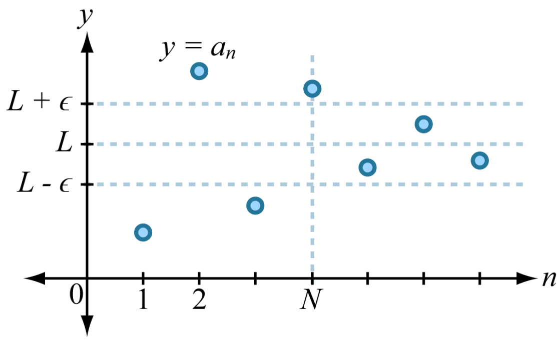
Types of series, convergence tests, and error bounds

Convergence of a sequence



We write

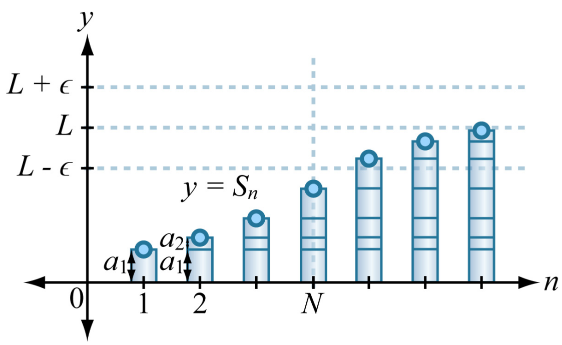
$$\lim_{n\to\infty} a_n = L$$

and say that the sequence (a_n) converges to L

to informally mean that the value a_n can be made to be arbitrarily close to L by requiring n to be sufficiently positively large

and to formally mean that so long as ϵ is a positive number, there exists a number N so that trapping integer n in $(N, +\infty)$ guarantees that a_n is trapped in $(L-\epsilon, L+\epsilon)$.

Convergence of a series



We write

$$S_n = \sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{\infty} a_i := \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} S_n = L$$

and say that the series $\sum_{i=1}^{\infty} a_i$ converges to L

to informally mean that the value S_n can be made to be arbitrarily close to L by requiring n to be sufficiently positively large

and to formally mean that so long as ϵ is a positive number, there exists a number N so that trapping integer n in $(N, +\infty)$ guarantees that S_n is trapped in $(L-\epsilon, L+\epsilon)$.

Finite number of terms does not affect convergence

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n\,, \quad N \in \mathbb{Z}^+$$

Convergence of a series relates to convergence of its terms

		$\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n \neq 0$	$\lim_{n\to\infty} a_n$ DNE
$\sum_{n=1}^{\infty} a_n$	Conv.	OK	No	No
	Div.	OK	OK	OK

Identities

Hypotheses

$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge

Conclusions

The following series (LHS) also converge and have the following values (RHS).

$$\sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n\,, \quad c \text{ a constant}$$

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n$$

Theorems are discussed in Stewart *Calculus* 4$^{\text{th}}$ ed. p. 743-745.

Types of series, convergence tests, and error bounds

Descriptions of tests/series	Conditions	Conclusions/formulas
Geometric $S = \sum_{n=0}^{\infty} ar^n$	$\lvert r\rvert < 1$	converges $S = \frac{a}{1-r}$
	$\lvert r\rvert \geq 1$	diverges
***p*-series** $S = \sum_{n=1}^{\infty} \frac{1}{n^p}$ $p > 0$	$p > 1$	converges
	$p \leq 1$	diverges
Integral test $S = \sum_{n=1}^{\infty} a_n$ $a_n = f(n)$ $f(n)$ is • a positive function • decreasing on $[1, \infty)$	$\int_1^{\infty} f(x)\,\mathrm{d}x \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$ $\int_1^{\infty} f(x)\,\mathrm{d}x \text{ diverges} \Leftrightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$ If $\sum_{n=1}^{\infty} a_n$ converges by the integral test, then $\int_{n+1}^{\infty} f(x)\,\mathrm{d}x \leq \underbrace{S - S_n}_{R_n} \leq \int_{n}^{\infty} f(x)\,\mathrm{d}x$	

Types of series, convergence tests, and error bounds

Descriptions of tests/series	Conditions	Conclusions/formulas
Comparison test $S_A = \sum_{n=1}^{\infty} a_n$ $S_B = \sum_{n=1}^{\infty} b_n$ Both a_n and $b_n > 0$	$\sum_{n=1}^{\infty} b_n$ converges $a_n \leq b_n$	$\sum_{n=1}^{\infty} a_n$ converges
	$\sum_{n=1}^{\infty} b_n$ diverges $a_n \geq b_n$	$\sum_{n=1}^{\infty} a_n$ diverges
Limit comparison test $S_A = \sum_{n=1}^{\infty} a_n$ $S_B = \sum_{n=1}^{\infty} b_n$ Both a_n and $b_n > 0$	$\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ where c is a (finite) number	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge - or - both diverge.

Types of series, convergence tests, and error bounds

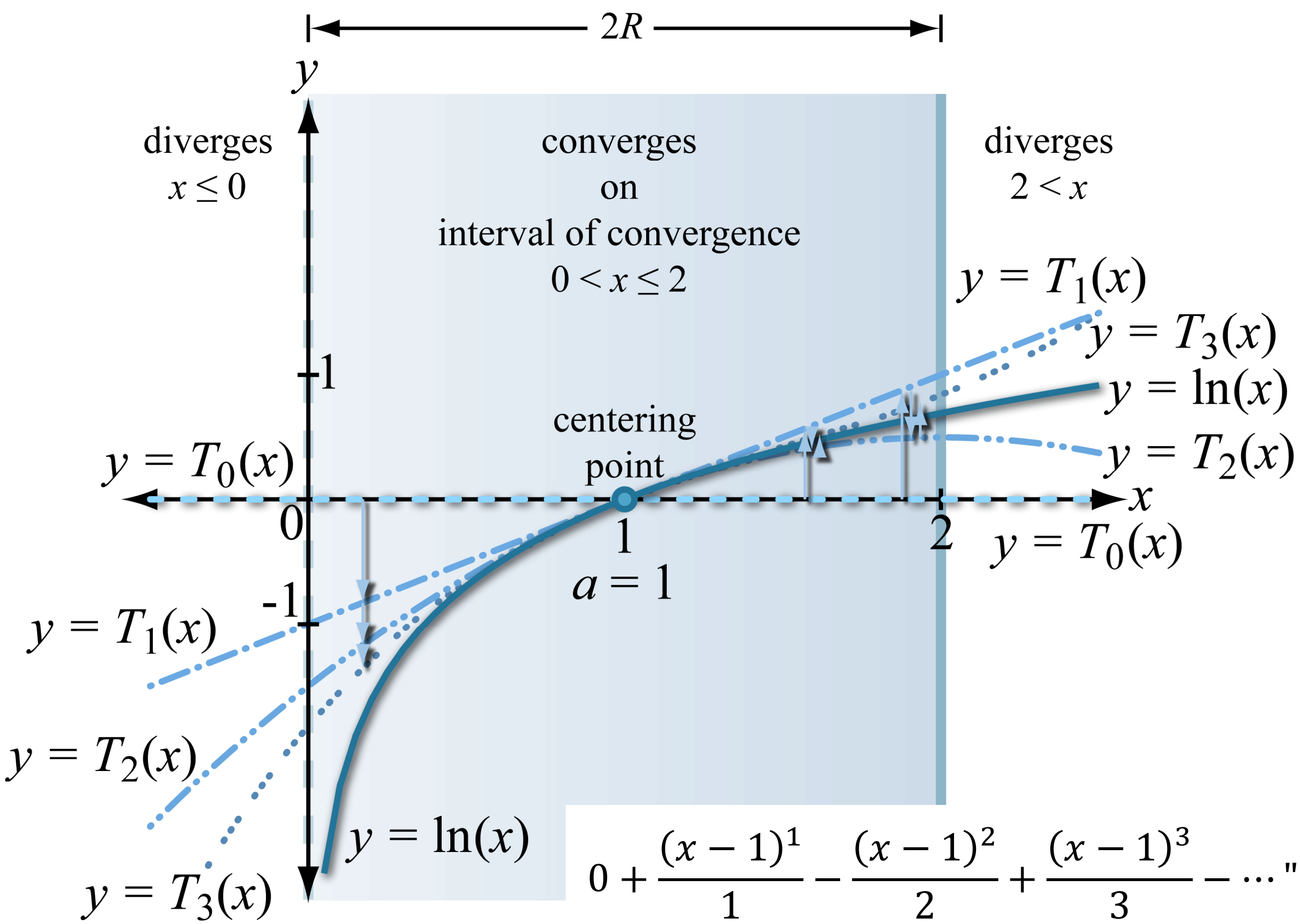
Terms in a sequence having different signs can partially cancel out. Terms having the same sign cannot cancel out.

If	$\sum_{n=1}^{\infty} a_n$	converges	and	$\sum_{n=1}^{\infty} \lvert a_n \rvert$	converges,	we say that	$\sum_{n=1}^{\infty} a_n$ converges absolutely.
If	$\sum_{n=1}^{\infty} a_n$	converges	and	$\sum_{n=1}^{\infty} \lvert a_n \rvert$	diverges,	we say that	$\sum_{n=1}^{\infty} a_n$ converges conditionally.
If	$\sum_{n=1}^{\infty} a_n$	diverges	and	$\sum_{n=1}^{\infty} \lvert a_n \rvert$	diverges,	we say that	$\sum_{n=1}^{\infty} a_n$ diverges.

Descriptions of tests/series	Conditions	Conclusions/formulas
Alternating series $S = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ $S = b_1 - b_2 + b_3 - b_4 + \cdots$ $b_n > 0$	$b_{n+1} \leq b_n \forall n$ $\lim_{n\to\infty} b_n = 0$	converges **“Forgotten” error bound** **!!!** You can use this even in problems where the wording makes you intuitively feel like studying the Lagrange error bound $\lvert R_n \rvert = \lvert S - S_n \rvert \leq b_{n+1}$
Ratio test **!!!** This is used a lot for finding the interiors of the intervals of convergence of power series $S = \sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L < 1$ L is a (finite) number	converges absolutely
	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L = 1$	inconclusive
	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = L > 1$	diverges
Root test $S = \sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = L < 1$	converges absolutely
	$\lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert} = \begin{cases} L > 1 \text{ or} \\ \infty \end{cases}$	diverges

Types of series, convergence tests, and error bounds

Power series $$S = \sum_{n=0}^{\infty} c_n (x-a)^n$$	3 possible behaviors: • converges only at $x = a$ (at centering/anchoring point) • converges $\forall x$ • converges on $\lvert x-a \rvert < R$ and diverges on $\lvert x-a \rvert > R$ for some $R > 0$ (check convergence at endpoints in specific situations)



$$0 + \frac{(x-1)^1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots \text{"} = \text{"} \ln(x)$$

Types of series, convergence tests, and error bounds

Taylor series

Series to be molded to imitate function	Function to be imitated	Recipe for imitation
$T(x) = c_0 + c_1(x-a)^1 + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$	$f(x)$	
Features to be molded	Features to be imitated	
$T^{(0)}(a) = c_0 + c_1(a-a)^1 + c_2(a-a)^2 + c_3(a-a)^3 + c_4(a-a)^4 + \cdots$	$f^{(0)}(a)$	
$T'(x) = 0 + c_1 + 2c_2(x-a)^1 + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$		
$T'(a) = 0 + c_1 + 2c_2(a-a)^1 + 3c_3(a-a)^2 + 4c_4(a-a)^3 + \cdots$	$f'(a)$	
$T''(x) = 0 + 0 + 2\cdot 1c_2 + 3\cdot 2c_3(x-a)^1 + 4\cdot 3c_4(x-a)^2 + \cdots$		
$T''(a) = 0 + 0 + 2\cdot 1c_2 + 3\cdot 2c_3(a-a)^1 + 4\cdot 3c_4(a-a)^2 + \cdots$	$f''(a)$	
$\vdots$	$\vdots$	
$T^{(n)}(a) = n!\,c_n$	$f^{(n)}(a)$	$c_n = \dfrac{f^{(n)}(a)}{n!}$

$$T(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If a Taylor series converges, it converges to the function from which it was constructed

If $\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} |f(x) - T_n(x)| = 0$ in an interval $|x-a| < R$, then $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(x)$ on $|x-a| < R$.

Lagrange error bound (**!!!** Do not have to use this if the alternating series error bound formula will work).

Hypotheses

- Considering a neighborhood $|x-a| \le d$ containing the centering point.
- In this neighborhood, the magnitude of the first neglected derivative of the true function is bounded, e.g. $\left|f^{(n+1)}(x)\right| \le M$, for some (finite) number M.

Conclusion

In this same neighborhood, $|x-a| \le d$, the error of the truncated Taylor polynomial of degree n is also bounded:

$$|R_n(x)| = |f(x) - T_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$