Calculus tricks with Taylor series: Representing a function

Sometimes you can write down an expression for the Taylor series of a function even while avoiding direct computation of the general formula for the *n*th derivative evaluated at the centering point.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Strategy 1: Differentiate power series of function that is antiderivative of function whose Taylor series is requested.

Example: Find the Taylor series expansion for $\frac{1}{(1-x)^2}$ with centering point at $x = 0$.

Function whose Taylor series is requested: $\frac{1}{(1-x)^2}$

Antiderivative of function	Power series representation of antiderivative (with centering point at $x = 0$).
$\dfrac{1}{1-x}$	$1 + x + x^2 + x^3 + x^4 + \cdots$

Differentiate antiderivative	Differentiate power series representation of derivative
$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = (-1)(1-x)^{-2}(-1)$ $= \frac{1}{(1-x)^2}$	$\frac{\mathrm{d}}{\mathrm{d}x}(1 + x + x^2 + x^3 + x^4 + \cdots)$ $= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots$

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

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Strategy 2: Integrate power series of function that is derivative of function whose Taylor series is requested.

Example: Find the Taylor series expansion for $\ln(x)$ with centering point at $x = 1$.

Function whose Taylor series is requested: $\ln(x)$

Derivative of function	Power series representation of derivative (with centering point at $x = 1$).
$\frac{1}{x}$	$\frac{1}{x} = \frac{1}{1+x-1}$ $= \frac{1}{1-[-(x-1)]}$ $= \frac{1}{1-[-(x-1)]}$ $= 1 + [-(x-1)] + [-(x-1)]^2 + \cdots$

Integrate derivative	Integrate power series representation of derivative
$\int_{x=x_i}^{x=x_f} \frac{1}{x}\,\mathrm{d}x = [\ln\lvert x\rvert]_{x=x_i}^{x=x_f}$ $= \ln\lvert x_f\rvert - \ln\lvert x_i\rvert$ $= \ln\left\lvert\frac{x_f}{x_i}\right\rvert$ For x_i and $x_f > 0$, $= \ln\left(\frac{x_f}{x_i}\right)$	$\int_{x=x_i}^{x=x_f} 1 + \underbrace{[-(x-1)]}_{u} + [-(x-1)]^2 + \cdots\ \mathrm{d}x$ $= \int_{x=x_i}^{x=x_f} 1 + u + u^2 + \cdots\ (-\mathrm{d}u)$ $= \left[-u - \frac{u^2}{2} - \frac{u^3}{3} - \cdots\right]_{x=x_i}^{x=x_f}$ $= \left[-[-(x-1)] - \frac{[-(x-1)]^2}{2} - \frac{[-(x-1)]^3}{3} - \cdots\right]_{x=x_i}^{x=x_f}$ $= \left[(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots\right]_{x=x_i}^{x=x_f}$

$$\ln\left(\frac{x_f}{x_i}\right) = \left[(x_f - 1) - \frac{(x_f-1)^2}{2} + \frac{(x_f-1)^3}{3} - \cdots\right] - \left[(x_i - 1) - \frac{(x_i-1)^2}{2} + \frac{(x_i-1)^3}{3} - \cdots\right]$$

Let $x_i = 1$ and $x_f =$ "x".

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$$