Calculus tricks with Taylor series: Representing a function

Sometimes you can write down an expression for the Taylor series of a function even while avoiding direct computation of the general formula for the *n*th derivative evaluated at the centering point.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Strategy 1: Differentiate power series of function that is antiderivative of function whose Taylor series is requested.

Example: Find the Taylor series expansion for $\frac{1}{(1-x)^2}$ with centering point at $x = 0$.

Function whose Taylor series is requested: $\frac{1}{(1-x)^2}$

Antiderivative of function	*might* be possible to represent as	Power series (with centering point at $x = 0$)	for some finite interval.
$\frac{1}{1-x}$		$1 + x + x^2 + x^3 + x^4 + \cdots$	

Differentiation of antiderivative	*might* …	Differentiation of power series	for … .
$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1-x}\right) = (-1)(1-x)^{-2}(-1)$ $= \frac{1}{(1-x)^2}$		$\frac{\mathrm{d}}{\mathrm{d}x}(1 + x + x^2 + x^3 + x^4 + \cdots)$ $= 0 + 1 + 2x + 3x^2 + 4x^3 + \cdots$	

Perhaps $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$ for some finite interval.

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Strategy 2: Integrate power series of function that is derivative of function whose Taylor series is requested.

Example: Find the Taylor series expansion for $\ln(x)$ with centering point at $x = 1$.

Function whose Taylor series is requested: $\ln(x)$

Derivative of function		Power series (with centering point at $x = 1$)	
$\frac{1}{x}$	*might* be possible to represent as	$\frac{1}{x} = \frac{1}{1+x-1} = \frac{1}{1-[-(x-1)]} = 1 + [-(x-1)] + [-(x-1)]^2 + \cdots$	for some finite interval.

Integration of derivative		Integration of power series	
$\int_{t=1}^{t=x} \frac{1}{t}\,\mathrm{d}t = [\ln\lvert t\rvert]_{t=1}^{t=x} = \ln\lvert x\rvert - \ln\lvert 1\rvert = \ln\left\lvert\frac{x}{1}\right\rvert$ For $x > 0$, $\int_{t=1}^{t=x} \frac{1}{t}\,\mathrm{d}t = \ln(x)$	…	$\int_{t=1}^{t=x} 1 + \underbrace{[-(t-1)]}_{u} + [-(t-1)]^2 + \cdots\,\mathrm{d}t = \int_{t=1}^{t=x} 1 + u + u^2 + \cdots\,(-\mathrm{d}u) = \left[-u - \frac{u^2}{2} - \frac{u^3}{3} - \cdots\right]_{t=1}^{t=x} = \left[-[-(t-1)] - \frac{[-(t-1)]^2}{2} - \frac{[-(t-1)]^3}{3} - \cdots\right]_{t=1}^{t=x}$ $\int_{t=1}^{t=x} \frac{1}{t}\,\mathrm{d}t = \left[(t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} - \cdots\right]_{t=1}^{t=x}$	…

Perhaps

$$\ln(x) = \left[(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots\right] - \left[(1-1) - \frac{(1-1)^2}{2} + \frac{(1-1)^3}{3} - \cdots\right]$$

for some finite interval.

Perhaps $\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots$ for some finite interval.