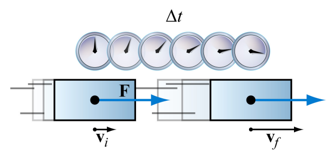
Newton’s second law can be re-expressed

$$\vec{\mathbf{a}} = \frac{\sum \vec{\mathbf{F}}}{m}$$

Impulse-momentum theorem (focus on force and time)



Algebra-based physics

Consider time-averaged force

$$\vec{\mathbf{a}}_{\text{AVG}} = \frac{\sum \vec{\mathbf{F}}_{\text{AVG}}}{m}$$

$$\frac{\sum \vec{\mathbf{F}}_{\text{AVG}}}{m} = \frac{\Delta \vec{\mathbf{v}}}{\Delta t}$$

$$\sum \vec{\mathbf{F}}_{\text{AVG}} \Delta t = m \Delta \vec{\mathbf{v}}$$

$$= m(\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i)$$

$$= m\vec{\mathbf{v}}_f - m\vec{\mathbf{v}}_i$$

$$\underbrace{\sum \vec{\mathbf{F}}_{\text{AVG}} \Delta t}_{\Delta \vec{\mathbf{J}}_{\Sigma \vec{\mathbf{F}}}} = \Delta\left(\underbrace{m\vec{\mathbf{v}}}_{\vec{\mathbf{p}}}\right)$$

Calculus-based physics

$$\frac{\sum \vec{\mathbf{F}}}{m} = \frac{d\vec{\mathbf{v}}}{dt}$$

$$\int_{t=t_i}^{t=t_f} \sum \vec{\mathbf{F}}\, dt = \int_{\vec{\mathbf{v}}=\vec{\mathbf{v}}_i}^{\vec{\mathbf{v}}=\vec{\mathbf{v}}_f} m\, d\vec{\mathbf{v}}$$

$$= m \int_{\vec{\mathbf{v}}=\vec{\mathbf{v}}_i}^{\vec{\mathbf{v}}=\vec{\mathbf{v}}_f} d\vec{\mathbf{v}}$$

$$= m(\vec{\mathbf{v}}_f - \vec{\mathbf{v}}_i)$$

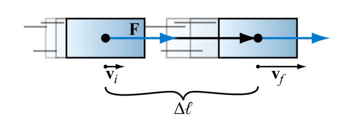
$$= m\vec{\mathbf{v}}_f - m\vec{\mathbf{v}}_i$$

$$\underbrace{\int_{t=t_i}^{t=t_f} \sum \vec{\mathbf{F}}\, dt}_{\Delta \vec{\mathbf{J}}_{\Sigma \vec{\mathbf{F}}}} = \Delta\left(\underbrace{m\vec{\mathbf{v}}}_{\vec{\mathbf{p}}}\right)$$

$$\Delta \vec{\mathbf{J}}_{\Sigma \vec{\mathbf{F}}} = \Delta \vec{\mathbf{p}}$$

$$\vec{\mathbf{p}}_i + \overbrace{\sum_{\text{F}} \Delta \vec{\mathbf{J}}_{\text{F}}}^{\Delta \vec{\mathbf{J}}_{\Sigma \vec{\mathbf{F}}}} = \vec{\mathbf{p}}_f$$

Work-energy theorem (focus on force and position)



Algebra-based physics

Consider uniform acceleration in 1-d

$$v_i^2 + 2a\Delta\ell = v_f^2$$

$$v_i^2 + 2\left(\frac{\sum F}{m}\right)\Delta\ell = v_f^2$$

$$\underbrace{\frac{1}{2}mv_i^2}_{KE_i} + \underbrace{\left(\sum F\right)\Delta\ell}_{\Delta W_{\Sigma \vec{\mathbf{F}}}} = \underbrace{\frac{1}{2}mv_f^2}_{KE_f}$$

Calculus-based physics

$$\frac{d\vec{\mathbf{v}}}{dt} = \frac{\sum \vec{\mathbf{F}}}{m}$$

$$\int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_i}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \sum \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{\ell}} = \int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_i}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} m \frac{d\vec{\mathbf{v}}}{dt} \cdot d\vec{\boldsymbol{\ell}}$$

$$= \int_{\vec{\mathbf{v}}=\vec{\mathbf{v}}_i}^{\vec{\mathbf{v}}=\vec{\mathbf{v}}_f} m\, d\vec{\mathbf{v}} \cdot \frac{d\vec{\boldsymbol{\ell}}}{dt}$$

$$= \int_{\vec{\mathbf{v}}=\vec{\mathbf{v}}_i}^{\vec{\mathbf{v}}=\vec{\mathbf{v}}_f} m\, d\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$$

$$= \int_{\vec{\mathbf{v}}=\vec{\mathbf{v}}_i}^{\vec{\mathbf{v}}=\vec{\mathbf{v}}_f} m \frac{d(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}})}{2}$$

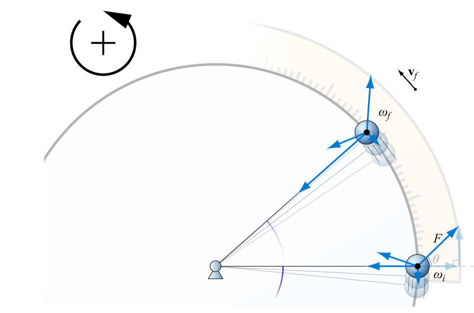
$$= \frac{1}{2}m \int_{v=v_i}^{v=v_f} d(v^2)$$

$$\underbrace{\int_{\vec{\mathbf{r}}=\vec{\mathbf{r}}_i}^{\vec{\mathbf{r}}=\vec{\mathbf{r}}_f} \sum \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{\ell}}}_{\Delta W_{\Sigma \vec{\mathbf{F}}}} = \underbrace{\frac{1}{2}mv_f^2}_{KE_f} - \underbrace{\frac{1}{2}mv_i^2}_{KE_i}$$

$$\Delta W_{\Sigma \vec{\mathbf{F}}} = \Delta KE$$

$$KE_i + \Delta W_{\Sigma \vec{\mathbf{F}}} = KE_f$$

Rotational formulation for Newton’s second law (change coordinate system)



$$a_{\text{IN}}(-\hat{\mathbf{r}}) + a_{\text{TAN}}\hat{\mathbf{t}} = \frac{\sum F_{\text{IN}}(-\hat{\mathbf{r}}) + \sum F_{\text{TAN}}\hat{\mathbf{t}}}{m}$$

$$a_{\text{IN}}(-\hat{\mathbf{r}}) + r\alpha\hat{\mathbf{t}} = \frac{\sum F_{\text{IN}}(-\hat{\mathbf{r}}) + \sum F_{\text{TAN}}\hat{\mathbf{t}}}{m}$$

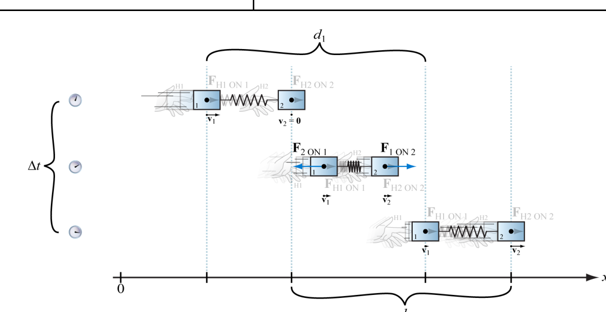
$$r\alpha = \frac{\sum F_{\text{TAN}}}{m}$$

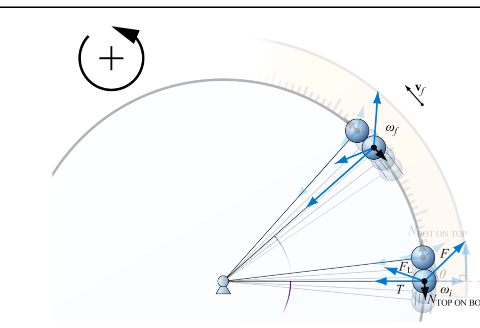
$$\alpha = \frac{\sum F_{\text{TAN}}}{mr}$$

$$\alpha = \frac{\sum rF_{\text{TAN}}}{mr^2}$$

$$\alpha = \frac{\overbrace{\sum rF \sin\theta}^{\tau}}{\underbrace{mr^2}_{I}}$$

$$\alpha = \frac{\sum \tau}{I}$$





Impulse-momentum: For system of N particles

$$\vec{\mathbf{p}}_{1,i} + \Delta \vec{\mathbf{J}}_{\Sigma \vec{\mathbf{F}}\ \text{ON 1}} = \vec{\mathbf{p}}_{1,f}$$

$$\vec{\mathbf{p}}_{2,i} + \Delta \vec{\mathbf{J}}_{\Sigma \vec{\mathbf{F}}\ \text{ON 2}} = \vec{\mathbf{p}}_{2,f}$$

$$\vdots$$

$$\vec{\mathbf{p}}_{N,i} + \Delta \vec{\mathbf{J}}_{\Sigma \vec{\mathbf{F}}\ \text{ON } N} = \vec{\mathbf{p}}_{N,f}$$

$$\Sigma \vec{\mathbf{P}}_i + \Sigma \Delta \vec{\mathbf{J}}_{\Sigma \vec{\mathbf{F}}} = \Sigma \vec{\mathbf{P}}_f$$

$$\Sigma \vec{\mathbf{P}}_i + \sum_{\substack{\text{EXT}\\ \text{ON SYS}}} \Delta \vec{\mathbf{J}} + \sum_{\text{INT}} \Delta \vec{\mathbf{J}} = \Sigma \vec{\mathbf{P}}_f$$

Internal Newton’s third law interaction force pairs have opposite directions. The magnitude vs. time plots for each force from a third law pair is identical. Thus, pairs of impulses corresponding to third law force pairs cancel out.

$$\Sigma \vec{\mathbf{P}}_i + \overbrace{\sum_{\substack{\text{EXT}\\ \text{ON SYS}}} \Delta \vec{\mathbf{J}}_{\text{F}}}^{\Delta \vec{\mathbf{J}}_{\Sigma \vec{\mathbf{F}},\text{EXT ON SYS}}} = \Sigma \vec{\mathbf{P}}_f$$

Work-energy: For system of N particles

$$KE_{1,i} + \Delta W_{\Sigma \vec{\mathbf{F}}\ \text{ON 1}} = KE_{1,f}$$

$$KE_{2,i} + \Delta W_{\Sigma \vec{\mathbf{F}}\ \text{ON 2}} = KE_{2,f}$$

$$\vdots$$

$$KE_{N,i} + \Delta W_{\Sigma \vec{\mathbf{F}}\ \text{ON } N} = KE_{N,f}$$

$$\Sigma KE_i + \Sigma \Delta W_{\Sigma \vec{\mathbf{F}}} = \Sigma KE_f$$

$$\Sigma KE_i + \sum_{\substack{\text{EXT}\\ \text{ON SYS}}} \Delta W + \sum_{\substack{\text{INT}\\ \text{CONS}}} \Delta W + \sum_{\substack{\text{INT}\\ \text{NC}}} \Delta W = \Sigma KE_f$$

Sums over internal works do not necessarily vanish. Even though forces occur in “third law” pairs, the corresponding pairs of works do not necessarily have equal magnitudes.

$$\Sigma KE_i + \sum_{\substack{\text{EXT}\\ \text{ON SYS}}} \Delta W - \sum_{\substack{\text{INT}\\ \text{CONS}}} \Delta U - \sum_{\substack{\text{INT}\\ \text{NC}}} \Delta U = \Sigma KE_f$$

Depending on how you do bookkeeping, the terms you write inside $\sum_{\substack{\text{INT}\\ \text{CONS}}} -\Delta U$ might not 1:1 correspond to the terms you write inside $\sum_{\substack{\text{INT}\\ \text{CONS}}} \Delta W$.

$$\Sigma KE_i + \sum_{\substack{\text{EXT}\\ \text{ON SYS}}} \Delta W - \sum_{\substack{\text{INT}\\ \text{CONS}}} \Delta U_{\text{G}} - \sum_{\substack{\text{INT}\\ \text{CONS}}} \Delta U_{\text{S}} - \sum_{\substack{\text{INT}\\ \text{NC}}} \Delta U = \Sigma KE_f$$

$$\overbrace{\Sigma KE_i + \Sigma U_{\text{G},i} + \Sigma U_{\text{S},i}}^{\Sigma ME_{\text{SYS},i}} + \sum_{\substack{\text{EXT}\\ \text{ON SYS}}} \Delta W_{\text{F}} = \overbrace{\Sigma KE_f + \Sigma U_{\text{G},f} + \Sigma U_{\text{S},f}}^{\Sigma ME_{\text{SYS},f}} + \Sigma \Delta U_{\text{INT}}$$

Rotational: For a rigid system of N particles, each particle has same α

$$I_1\alpha = \sum_{\text{ON 1}} \tau$$

$$I_2\alpha = \sum_{\text{ON 2}} \tau$$

$$\vdots$$

$$I_N\alpha = \sum_{\text{ON } N} \tau$$

$$(\Sigma I)\alpha = \sum_{\text{ON 1}} \tau + \sum_{\text{ON 2}} \tau + \cdots + \sum_{\text{ON } N} \tau$$

$$(\Sigma I)\alpha = \sum_{\substack{2,\ldots,N\\ \text{ON 1}}} \tau + \sum_{\substack{\text{EXT}\\ \text{ON 1}}} \tau + \sum_{\substack{1,3,\ldots,N\\ \text{ON 2}}} \tau + \sum_{\substack{\text{EXT}\\ \text{ON 2}}} \tau + \cdots + \sum_{\substack{1,\ldots,N-1\\ \text{ON } N}} \tau + \sum_{\substack{\text{EXT}\\ \text{ON } N}} \tau$$

There is no known force law that generates Newton’s third law interaction force pairs that do not share a common line of action. Using this fact and the above illustration, we can argue that all internal torque pairs cancel.

$$(\Sigma I)\alpha = \sum_{\substack{\text{EXT}\\ \text{ON SYS}}} \tau$$

$$\alpha = \frac{\sum \tau}{I}$$

Focus on torque and time	Focus on torque and angular position
$L := I\omega$	$KE_{\text{RIGID WITH FIXED A.O.R.}} = \frac{1}{2} I_{\substack{\text{ABOUT}\\ \text{FIXED}\\ \text{SKEWER}}} \omega^2$ $\Delta W_{\Sigma \vec{\mathbf{F}}} = \left(\sum_{\substack{\text{EXT}\\ \text{ON SYS}}} \tau_{\text{AVG}}\right)\Delta\theta$
$\Sigma L_i + \left(\sum_{\substack{\text{EXT}\\ \text{ON SYS}}} \tau_{\text{AVG}}\right)\Delta t = \Sigma L_f$	$KE_{\text{R.W.F.A.O.R.},i} + \Delta W_{\Sigma \vec{\mathbf{F}}} = KE_{\text{R.W.F.A.O.R.},f}$