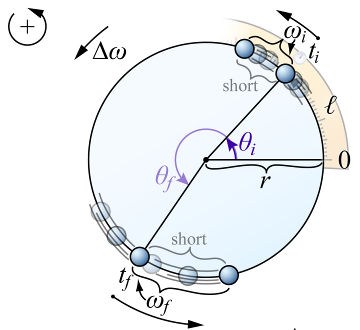
Rotational kinematics and dynamics (calculus-based physics)

Kinematics

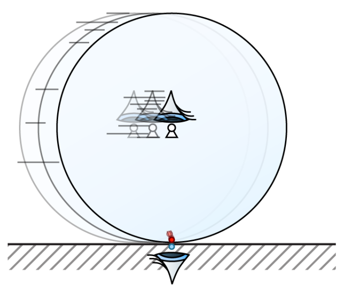


Definitions

Angular	Tangential
$\theta := \dfrac{\ell}{r}$	$\Delta\ell = r\Delta\theta$
$\omega := \dfrac{d\theta}{dt}$	$v_{\text{TAN}} = r\omega$
$\alpha := \dfrac{d\omega}{dt}$	$a_{\text{TAN}} = r\alpha$

Relationships

If non-slip
$\Delta x_{\text{A.O.R.}} = \Delta\ell$
$v_{\text{A.O.R.}} = v_{\text{TAN}}$
$a_{\text{A.O.R.}} = a_{\text{TAN}}$



Explanation of no-slip condition

$$v_{\substack{\text{FLOOR}\\\text{REL}\\\text{A.O.R.}}} = v_{\text{TAN}}$$

$$v_{\substack{\text{A.O.R.}\\\text{REL}\\\text{FLOOR}}} = v_{\substack{\text{FLOOR}\\\text{REL}\\\text{A.O.R.}}}$$

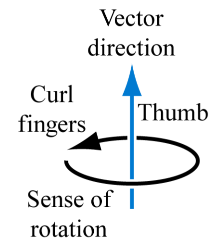
Relationships for UαM

$\theta_i + \omega_{\text{AVG}}\,\Delta t = \theta_f$	α
$\omega_i + \alpha_{\text{AVG}}\,\Delta t = \omega_f$	θ
$\omega_{\text{AVG}} = \dfrac{\omega_i + \omega_f}{2}$	t, θ, α
$\theta_i + \omega_i\Delta t + \dfrac{1}{2}\alpha\Delta t^2 = \theta_f$	
$\omega_i^2 + 2\alpha\Delta\theta = \omega_f^2$	t

Dynamics

Rotational vectors

RHR

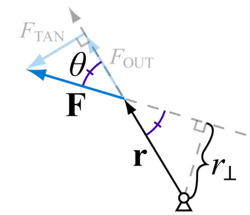


Torque

$$\vec{\boldsymbol{\tau}}_{\text{F}} := \vec{\mathbf{r}} \times \vec{\mathbf{F}}$$

$$\tau_{\text{F}} = r_\perp F = (r\sin\theta)F$$

$$\vec{\boldsymbol{\alpha}} = \frac{\sum\vec{\boldsymbol{\tau}}}{I}$$



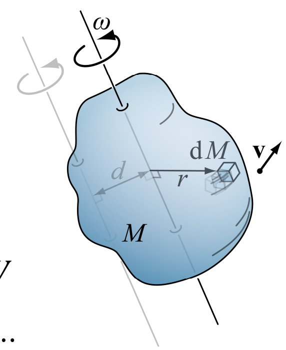
Rotational inertia

$$I_{\substack{\text{RIGID SET}\\\text{OF}\\\text{PARTICLES}}} := \sum_i \Delta M_i r_i^2$$

$$I_{\substack{\text{RIGID}\\\text{CONTINUOUS}\\\text{MASS DISTR.}}} := \int r^2\,dM = \int r^2\rho\,dV$$

$$I_{\substack{\text{RIGID}\\\text{PARTS}}} = I_1 + I_2 + I_3 + \cdots$$

$$I_\parallel = I_{\text{CM}} + Md^2$$



Summing torques

1. Draw spatially-extended **free-body diagram** with the **tail** of each force vector anchored at its **point of application**.
2. Draw $+x$ and $+y$ directions, **axis of rotation**, and positive **sense of rotation**.
3. Fill in $\sum\tau = I\alpha$, determining the **sign** of each τ by considering whether each force, in isolation, would spin up the object in the ccw or cw direction.

Conservation laws

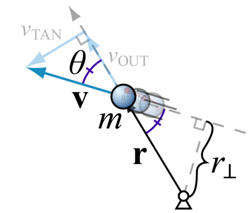
Angular momentum

$$\vec{\mathbf{L}}_{\text{PARTICLES}} := \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{p}}_i$$

$$L_{\text{PARTICLE}} = mvr_\perp = (r\sin\theta)p$$

$$\vec{\mathbf{L}}_{\text{RIGID}} = I_{\substack{\text{ABOUT}\\\text{FIXED}\\\text{SKEWER}}}\vec{\boldsymbol{\omega}}$$

$$\vec{\mathbf{L}}_{\text{RIGID}} = \vec{\mathbf{L}}_{\substack{\text{C.O.M.}\\\text{ORBITS}\\\text{ORIGIN}}} + \vec{\mathbf{L}}_{\substack{\text{SPIN}\\\text{ABOUT}\\\text{C.O.M.}}}$$



$$\frac{d\vec{\mathbf{L}}}{dt} = \Sigma\vec{\boldsymbol{\tau}}$$

$$\Sigma\vec{\mathbf{L}}_i + \int_{t=t_i}^{t=t_f}\left(\sum_{\substack{\text{EXT}\\\text{ON SYS}}}\vec{\boldsymbol{\tau}}\right)dt = \Sigma\vec{\mathbf{L}}_f$$

Summing angular momenta

1. Illustrate **before** and **after** situations.
2. Draw **axis of rotation**.
3. Draw positive **sense of rotation**.
4. Determine **sign** of each object's L by determining whether rotation is ccw or cw.

Energy

$$K_{\text{PARTICLES}} := \sum_i \frac{1}{2}\Delta M_i v_i^2$$

$$K_{\substack{\text{CONTINUOUS}\\\text{MASS DISTR.}}} := \int \frac{1}{2}v^2\,dM$$

$$K_{\text{RIGID}} = \frac{1}{2}I_{\substack{\text{ABOUT}\\\text{FIXED}\\\text{AXIS}}}\omega^2 \qquad \Delta W_{\tau_{\text{F}}} = \int_{\theta=\theta_i}^{\theta=\theta_f}\tau_{\text{F}}\,d\theta$$

$$K_{\text{RIGID}} = \frac{1}{2}Mv_{\text{C.O.M.}}^2 + \frac{1}{2}I_{\substack{\text{ABOUT}\\\text{C.O.M.}}}\omega^2$$