Heuristic for writing a proof: ComMisERGATE

Complain – is something **mis**sing? Is there something **e**xtra you wish were **r**emoved?
Immediate **g**oal – What do you want to do about the problem you described?
Allowed step – On your reference sheet(s), identify a piece of allowed knowledge that can help you to achieve your goal
Try – Try to apply the allowed knowledge
Evaluate – Did applying the allowed knowledge help you to achieve your goal?

<table>
<tr><th>Steps (observe numbered order)</th><th>Example (“given → to show” demonstrated; OK to work from either end/both ends)</th></tr>
<tr><td>0. Prepare a reference sheet that displays all allowed/required first principles (axioms/postulates, definitions, and theorems) in a format that is easy to navigate at a glance.</td><td>If you are starting to read a chapter in linear algebra that introduces vector spaces, your reference sheet should have the definition of a vector space, which includes 10 properties associated with the vector multiplication $\oplus$ and scalar multiplication $\odot$ operations.
It is also useful to include tips from example proofs in your reference sheet. For example, it is useful to know that proofs of uniqueness can sometimes be completed by writing down two copies of a relevant definition and then showing that the two differently-labeled objects in the two equations must be the same object.</td></tr>
<tr><td></td><td></td></tr>
<tr><td>1. Copy the question</td><td>Demonstrate that the additive identity in a vector space $\mathbb{V}$ is unique.</td></tr>
<tr><td>2. Translate givens</td><td>Vector space $\mathbb{V}$</td></tr>
<tr><td></td><td></td></tr>
<tr><td>4. Complain about a feature of the work you have obtained so far: “_____ is missing or ______ is extra and should be removed.”</td><td>At this point, we haven’t written down anything except for the given statement. One problem is that, currently, we don’t have any mention of the $\vec{\mathbf{0}}_1$ or $\vec{\mathbf{0}}_2$ that appear in the translation of what is to be demonstrated.</td></tr>
<tr><td>5. Immediate goal: What do you want to do about this problem?</td><td>Find some way to mention $\vec{\mathbf{0}}_1$ or $\vec{\mathbf{0}}_2$</td></tr>
<tr><td>6. Browse your reference sheet and suggest an allowed tool/approach</td><td>Name drop $\vec{\mathbf{0}}_1$ and $\vec{\mathbf{0}}_2$ using equations that define “additive identity”</td></tr>
<tr><td>7. Try it</td><td>One of the defining properties of a vector space is that there is an additive identity. Suppose that in the given vector space $\mathbb{V}$, two additive identities can be found.
$\forall\vec{\mathbf{u}} \in \mathbb{V}$, $\vec{\mathbf{u}} \oplus \vec{\mathbf{0}}_1 = \vec{\mathbf{u}}$ and $\vec{\mathbf{u}} \oplus \vec{\mathbf{0}}_2 = \vec{\mathbf{u}}$</td></tr>
<tr><td>8. Evaluate: Did that help?</td><td>Yes. We have now mentioned $\vec{\mathbf{0}}_1$ and $\vec{\mathbf{0}}_2$. Yay!</td></tr>
<tr><td></td><td></td></tr>
<tr><td>9. Complain about a feature of the work you have obtained so far: “_____ is missing or ______ is extra and should be removed.”</td><td>$\vec{\mathbf{0}}_1$ and $\vec{\mathbf{0}}_2$ are in two different equations, rather than in a single equation.</td></tr>
<tr><td>10. Immediate goal: What do you want to do about this problem?</td><td>Make $\vec{\mathbf{0}}_1$ and $\vec{\mathbf{0}}_2$ appear together in a single equation</td></tr>
<tr><td>11. Browse your reference sheet and suggest an allowed tool/approach</td><td>Set the LHSs equal.</td></tr>
<tr><td>12. Try it</td><td>Both LHSs in $\vec{\mathbf{u}} \oplus \vec{\mathbf{0}}_1 = \vec{\mathbf{u}}$ and $\vec{\mathbf{u}} \oplus \vec{\mathbf{0}}_2 = \vec{\mathbf{u}}$
equal the same RHSs, so the LHSs equal each other (transitive/substitution)
$\vec{\mathbf{u}} \oplus \vec{\mathbf{0}}_1 = \vec{\mathbf{u}} \oplus \vec{\mathbf{0}}_2$</td></tr>
<tr><td>13. Evaluate: Did that help?</td><td>Yes. $\vec{\mathbf{0}}_1$ and $\vec{\mathbf{0}}_2$ now appear together in a single equation. Yay!</td></tr>
<tr><td></td><td></td></tr>
<tr><td>14. Complain about a feature of the work you have obtained so far: “_____ is missing or ______ is extra and should be removed.”</td><td>I don’t want the vector $\vec{\mathbf{u}}$ to show up in my final answer. Boo!</td></tr>
<tr><td>15. Immediate goal: What do you want to do about this problem?</td><td>Get rid of the vector $\vec{\mathbf{u}}$.</td></tr>
<tr><td>16. Browse your reference sheet and suggest an allowed tool/approach</td><td>Use the additive inverse of $\vec{\mathbf{u}}$</td></tr>
<tr><td>17. Try it</td><td>One of the defining properties of a vector space is that each vector in it has an additive inverse. For vector $\vec{\mathbf{u}}$, an additive inverse is labeled $-\vec{\mathbf{u}}$ and defined by
$\vec{\mathbf{u}} \oplus -\vec{\mathbf{u}} = \vec{\mathbf{0}}$
<table>
<tr><td>$\vec{\mathbf{u}} \oplus \vec{\mathbf{0}}_1 = \vec{\mathbf{u}} \oplus \vec{\mathbf{0}}_2$
$-\vec{\mathbf{u}} \oplus (\vec{\mathbf{u}} \oplus \vec{\mathbf{0}}_1) = -\vec{\mathbf{u}} \oplus (\vec{\mathbf{u}} \oplus \vec{\mathbf{0}}_2)$</td><td>Add $-\vec{\mathbf{u}}$ to both sides</td></tr>
<tr><td>$(-\vec{\mathbf{u}} \oplus \vec{\mathbf{u}}) \oplus \vec{\mathbf{0}}_1 = (-\vec{\mathbf{u}} \oplus \vec{\mathbf{u}}) \oplus \vec{\mathbf{0}}_2$</td><td>Vector addition is associative (vector space property)</td></tr>
<tr><td>$(\vec{\mathbf{u}} \oplus -\vec{\mathbf{u}}) \oplus \vec{\mathbf{0}}_1 = (\vec{\mathbf{u}} \oplus -\vec{\mathbf{u}}) \oplus \vec{\mathbf{0}}_2$</td><td>Vector addition is commutative (vector space property)</td></tr>
<tr><td>$\vec{\mathbf{0}}_{\text{MAYBE NOT SAME AS 1 OR 2}} \oplus \vec{\mathbf{0}}_1$
$= \vec{\mathbf{0}}_{\text{MAYBE NOT SAME AS 1 OR 2}} \oplus \vec{\mathbf{0}}_2$</td><td>Definition of additive inverse</td></tr>
<tr><td>$\vec{\mathbf{0}}_1 \oplus \vec{\mathbf{0}}_{\text{MAYBE NOT SAME AS 1 OR 2}}$
$= \vec{\mathbf{0}}_2 \oplus \vec{\mathbf{0}}_{\text{MAYBE NOT SAME AS 1 OR 2}}$</td><td>Vector addition is commutative (vector space property)</td></tr>
<tr><td>$\vec{\mathbf{0}}_1 = \vec{\mathbf{0}}_2$</td><td>Definition of additive inverse</td></tr>
</table></td></tr>
<tr><td>18. Evaluate: Did that help?</td><td>Yes, we finally got rid of the $\vec{\mathbf{u}}$. We got the final result we wanted. Yay!</td></tr>
<tr><td></td><td></td></tr>
<tr><td>3. Translate what is to be demonstrated</td><td>(Remembering the “tip” in the reference sheet, we can translate the statement to be proved in the following way):
For any $\vec{\mathbf{0}}_1$ and $\vec{\mathbf{0}}_2$ that are both additive identities, $\vec{\mathbf{0}}_1 = \vec{\mathbf{0}}_2$</td></tr>
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