

NEUROTIC NITPICKING

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Math and science skills
for central Jersey's strained high schoolers

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Problem types	Activity category	Scaffold	Pg.	Related topics	References
Test experiment	Inquire (reject candidate models)	STEAM	2	inquiry-based learning	Etkina <i>et al.</i> islephysics.net
				Modeling Instruction™ patternicity	AMTA modelinginstruction.org Shermer 2008 doi.org/10.1038/scientificamerican1208-48
Word problems	Read & write notes & questions	SCAN	3	multiple representations	Larkin, McDermott, Simon, and Simon 1980 doi.org/10.1126/science.208.4450.1335 van Heuvelen 1991 doi.org/10.1119/1.16668
				idea distinct from name	Arons 1984 doi.org/10.1119/1.2341444
Algebra problems Geometry proofs Physics & chemistry problems	Develop arguments	STEP	4	formal arguments	Toulmin 2003 doi.org/10.1017/CBO9780511840005 McNeill, Lizotte, Krajcik, and Marx 2006 doi.org/10.1207/s15327809jls1502_1
				chains of reasoning	Speirs, Ferm, Stetzer, and Lindsey 2016 doi.org/10.1119/perc.2016.pr.077
Math and science	Narrate arguments	TELL	8, 10		(Same as for STEP)
Physics and chemistry problems	Spider-web thinking (track quantities and governing and constitutive relationships that simultaneously constrain them)	MAP	9	novice and expert problem-solving, production schemas	Chi, Feltovich, and Glaser 1981 doi.org/10.1207/s15516709cog0502_2
				Simultaneously diagramming governing and constitutive relationships	Rosengrant 2011 doi.org/10.1119/1.3527754

Lesson 1: Carry out **inquiry** (reject candidate models) using **STEAM**.

Example 1.1: What is $\sqrt{x^2}$? Alice proposes that $\sqrt{x^2} = x$, and Beth propose that $\sqrt{x^2} = |x|$.

	<u>Shiftable input(s)</u>	<u>Theoretical Expectations</u>		<u>Actual results and Measurements</u>
	x	By Alice's model $\sqrt{x^2} = x$,	By Beth's model $\sqrt{x^2} = x $,	$\sqrt{x^2}$
1.	-2	-2 😞	2	$\sqrt{(-2)^2} = \sqrt{4} = 2$
2.	-1	-1 😞	1	$\sqrt{(-1)^2} = \sqrt{1} = 1$
3.	0	0	0	$\sqrt{(0)^2} = \sqrt{0} = 0$
4.	1	1	1	$\sqrt{(1)^2} = \sqrt{1} = 1$
5.	2	2	2	$\sqrt{(2)^2} = \sqrt{4} = 2$

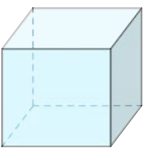
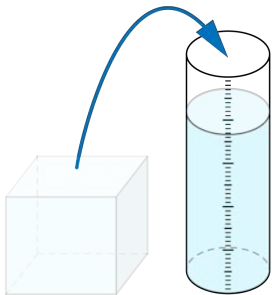
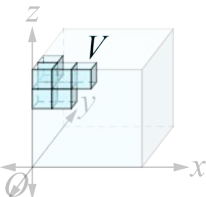
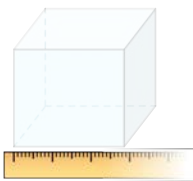
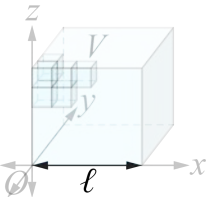
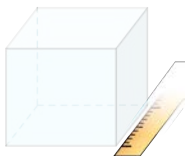
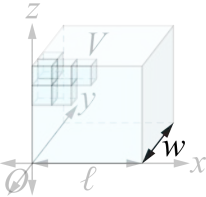
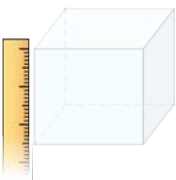
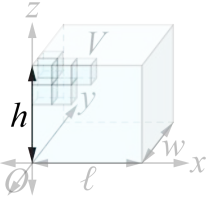
In the STEAM table, some of Alice's predictions are **inconsistent** with actual values (when x is negative), so **reject** Alice's model, $\sqrt{x^2} = x$.

In the STEAM table, all of Beth's predictions are **consistent** with actual values, so Beth's model $\sqrt{x^2} = |x|$ **remains in play**.

The STEAM table alone doesn't allow Beth's model to be "ruled in" since the STEAM table doesn't preclude the possibility of additional values of the shiftable input x resulting in values of $\sqrt{x^2}$ that turn out to fail to equal $|x|$.

Lesson 2: Read and write notes and questions using SCAN.

Example 2.1: Translate: “A rectangular parcel of water has a volume equaling the parcel’s length times the parcel’s width times the parcel’s height.”

	<u>Short phrase</u>	<u>Cartoon</u>	<u>Axes and plots</u>	<u>Notation</u>
1.	A rectangular parcel of water	Semi-transparent rectangular prism 		
2.	has a volume	Arrow indicating water can be poured from rectangular container into graduated cylinder used to measure volume 	Cartesian axis system implying scale for representative unit cubes within parcel 	Let $V =$ Volume
3.	equaling		Optional: Number line with two arrows pointing from the origin to same tickmark	$V =$
4.	the parcel’s length	Optional: Ruler used to measure length 	Double-headed arrow indicating size of an edge parallel to x -axis 	Let $\ell =$ width $V = \ell$
5.	times		Optional: Area model of multiplication	$V = \ell \cdot$
6.	the parcel’s width	Optional: Ruler used to measure width 	Double-headed arrow indicating size of an edge parallel to y -axis 	Let $w =$ width $V = \ell \cdot w$
7.	times		Optional: Area model of multiplication	$V = \ell \cdot w \cdot$
8.	the parcel’s height.	Optional: Ruler used to measure height 	Double-headed arrow indicating size of an edge parallel to z -axis 	Let $h =$ height $V = \ell \cdot w \cdot h$

Lesson 3: Develop arguments using STEP.

Example 3.1: Solve for x . Show each step that can't be accomplished on a basic four-function calculator.

$$4x - 2 = 98$$

	<u>Statement</u>	<u>Tool</u>	<u>Equivalent parts (of statement & tool)</u>	<u>Populated tool</u>
1.	$4x - 2 = 98$	$a = b$ \Downarrow $a + c = b + c$	$a = 4x - 2$ $b = 98$ $c = 2$	$4x - 2 = 98$ \Downarrow $4x - 2 + 2 = 98 + 2$
2.	$4x + 0 = 100$	$a \leftrightarrow a + 0 \leftrightarrow 0 + a$	$a = 4x$	$4x \leftrightarrow 4x + 0 \leftrightarrow 0 + 4x$
3.	$4x = 100$	$a = b$ \Downarrow $\frac{a}{c} = \frac{b}{c}, c \neq 0$	$a = 4x$ $b = 100$ $c = 4$	$4x = 100$ \Downarrow $\frac{4x}{4} = \frac{100}{4}, 4 \neq 0$
4.	$\frac{4x}{4} = 25$	$a \leftrightarrow a \cdot 1 \leftrightarrow 1 \cdot a$	$a = 4$	$4 \leftrightarrow 4 \cdot 1 \leftrightarrow 1 \cdot 4$
5.	$\frac{4x}{4 \cdot 1} = 25$	$\frac{ab}{cd} \leftrightarrow \frac{a b}{c d}$	$a = 4$ $b = x$ $c = 4$ $d = 1$	$\frac{4x}{4 \cdot 1} \leftrightarrow \frac{4x}{4 \cdot 1}$
6.	$\frac{4x}{4 \cdot 1} = 25$	$\frac{a}{a}, a \neq 0 \leftrightarrow 1$	$a = 4$	$\frac{4}{4}, 4 \neq 0 \leftrightarrow 1$
7.	$1 \cdot \frac{x}{1} = 25$	$\frac{a}{1} \leftrightarrow a$	$a = x$	$\frac{x}{1} \leftrightarrow x$
8.	$1 \cdot x = 25$	$a \cdot 1 \leftrightarrow 1 \cdot a \leftrightarrow a$	$a = x$	$x \cdot 1 \leftrightarrow 1 \cdot x \leftrightarrow x$
9.	$x = 25$			

To fill in the **Tool** column, use the basic algebra tools or basic pre-algebra tools on the following pages.

Basic algebra tools

For exercises, see corresponding “EA” sections in Marecek *et al.*, *Elementary Algebra 2e*, available for free through a CC BY license at openstax.org/details/books/elementary-algebra-2e

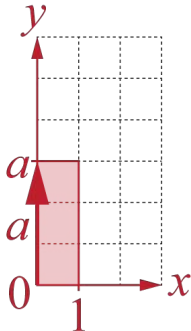
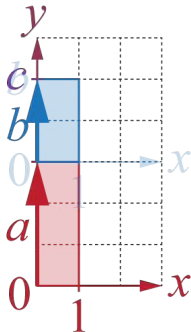
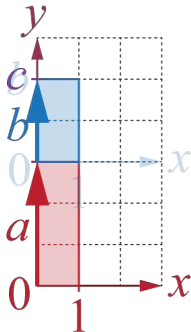
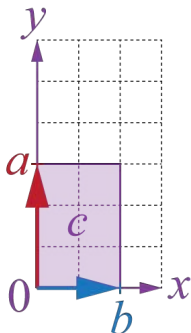
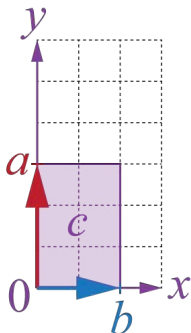
EA	Name	Tool(s)
1.9	A. Division by zero is undefined	Any denominator or factor in denominator = 0 ↓ Discard template
1.5	B. Product of fractions	$\frac{a}{b} \cdot \frac{c}{d} \leftrightarrow \frac{ac}{bd}$
1.5	C. Canceling factors in numerator & denominator	$\frac{a \cdot c}{b \cdot c} \leftrightarrow \frac{a}{b}$ $\frac{c \cdot a}{c \cdot b} \leftrightarrow \frac{a}{b}$
1.5	D. Quotient of fractions	$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} \leftrightarrow \frac{a}{b} \cdot \frac{d}{c}$
1.9	E. Commutative Property of Addition	$a + b$ ↓ $b + a$
1.9	F. Commutative Property of Multiplication	$a \cdot b$ ↓ $b \cdot a$
1.9	G. Associative Property of Addition	$a + b + c$ ↓ $(a + b) + c$ ↓ $a + (b + c)$
1.9	H. Associative Property of Multiplication	$a \cdot b \cdot c$ ↓ $(a \cdot b) \cdot c$ ↓ $a \cdot (b \cdot c)$

EA	Name	Tool(s)									
		$a \cdot (b + c)$ $(b + c) \cdot a$ ↓ ↓ $a \cdot b + a \cdot c$ $b \cdot a + c \cdot a$									
		$a \cdot (b - c)$ $(b - c) \cdot a$ ↓ ↓ $a \cdot b - a \cdot c$ $b \cdot a - c \cdot a$									
1.9	I. Distributive Property	$(a + b) \cdot (c + d)$ ↓ <table style="border-collapse: collapse; margin: auto;"> <tr> <td style="border-right: 1px solid black; border-bottom: 1px solid black; padding: 5px;"></td> <td style="border-bottom: 1px solid black; padding: 5px; text-align: center;">c</td> <td style="border-bottom: 1px solid black; padding: 5px; text-align: center;">d</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px; text-align: center;">a</td> <td style="padding: 5px; text-align: center;">$a \cdot c$</td> <td style="padding: 5px; text-align: center;">$a \cdot d$</td> </tr> <tr> <td style="border-right: 1px solid black; padding: 5px; text-align: center;">b</td> <td style="padding: 5px; text-align: center;">$b \cdot c$</td> <td style="padding: 5px; text-align: center;">$b \cdot d$</td> </tr> </table> $a \cdot c + b \cdot c + a \cdot d + b \cdot d$		c	d	a	$a \cdot c$	$a \cdot d$	b	$b \cdot c$	$b \cdot d$
	c	d									
a	$a \cdot c$	$a \cdot d$									
b	$b \cdot c$	$b \cdot d$									
1.9	J. Identity Property of Addition	$a + 0 \leftrightarrow 0 + a \leftrightarrow a$									
1.9	K. Identity Property of Multiplication	$a \cdot 1 \leftrightarrow 1 \cdot a \leftrightarrow a$									
1.9	L. Identity Property of Division	$\frac{a}{1} \leftrightarrow a$									
1.9	M. Adding additive inverse is equivalent to subtracting	$a + (-b)$ ↓ $a - b$									
1.9	N. Inverse Property of Addition	$a + (-a) \leftrightarrow a - a \leftrightarrow 0$									
1.5 & 1.9	O. Inverse Property of Multiplication (and the related One Property of Division)	$a \cdot \frac{1}{a}, a \neq 0 \leftrightarrow \frac{a}{a}, a \neq 0 \leftrightarrow 1$									

EA	Name	Tool(s)
1.9	P. Multiplication by zero	$a \cdot 0 \leftrightarrow 0 \cdot a \leftrightarrow 0$
1.9	Q. Division involving zero	$\frac{0}{a} \leftrightarrow 0$ See "Division by zero is undefined"
2.1	R. Addition Property of Equality	$a = b$ \downarrow $a + c = b + c$
2.1	S. Subtraction Property of Equality	$a = b$ \downarrow $a - c = b - c$
2.2	T. Multiplication Property of Equality	$a = b$ \downarrow $a \cdot c = b \cdot c$
2.2	U. Division Property of Equality	$a = b$ \downarrow $\frac{a}{c} = \frac{b}{c}$

EA	Name	Tool(s)
6.2	V. Exponent notation	$a^2 \leftrightarrow a \cdot a$ $a^n \leftrightarrow \underbrace{a \cdot a \cdots a}_{n \text{ copies}}$
7.6	W. Zero-Product Property	$a \cdot b = 0$ \downarrow $a = 0 \text{ or } b = 0$
9.1/9.7	X. Root notation	$\sqrt[n]{a} = b, \text{ even } n$ $\sqrt[n]{a} = b, \text{ odd } n$ \updownarrow \updownarrow $b^n = a, b \geq 0, \text{ even } n$ $b^n = a, \text{ odd } n$
9.8*	Y. A correction to the Power Property	$\sqrt{a^2} \leftrightarrow a $
10.3	Z. Quadratic Formula	$ax^2 + bx + c = 0$ \updownarrow $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Basic pre-algebra tools

	Name	Numeric example	Tool		Gridding steps
			Algebraic	Gridded	
A.	(Real) number	3	a		<ol style="list-style-type: none"> 1. Draw vertical arrow from origin (0) to y-tick labeled a. Label arrow a. 2. Regard vertical arrow as left edge of a rectangle of width 1. 3. Vertical arrow and rectangle represent the number a individually and together.
B.	Addition	$3 + 2 = 5$	$a + b = c$		<ol style="list-style-type: none"> 1. Represent the number a. 2. Regard arrowhead of arrow for a as origin of a new axis system. 3. Use new axis system to represent the number b. 4. Use original axes to read off y-tick, c, of arrowhead you just drew and complete sketch of the number c.
C.	Subtraction	$2 = 5 - 3$	$b = c - a$		<ol style="list-style-type: none"> 1. Represent number a. Ignore rectangle, if desired. 2. Draw horizontal arrow from origin (0) to x-tick labeled b. 3. Draw rectangle using vertical arrow for a and horizontal arrow for b as sides. Label signed number of unit squares, c, rectangle covers. 4. A rectangle in Quadrant I or Quadrant III has a positive signed number of unit squares. A rectangle in Quadrant II or Quadrant IV has a negative signed number of unit squares.
E.	Multiplication	$(3)(2) = 6$	$(a)(b) = c$		<ol style="list-style-type: none"> 1. Represent number a. Ignore rectangle, if desired. 2. Draw horizontal arrow from origin (0) to x-tick labeled b. 3. Draw rectangle using vertical arrow for a and horizontal arrow for b as sides. Label signed number of unit squares, c, rectangle covers. 4. A rectangle in Quadrant I or Quadrant III has a positive signed number of unit squares. A rectangle in Quadrant II or Quadrant IV has a negative signed number of unit squares.
F.	Division	$2 = \frac{6}{3}$	$b = \frac{c}{a}, a \neq 0$		<ol style="list-style-type: none"> 1. Represent number a. Ignore rectangle, if desired. 2. Draw horizontal arrow from origin (0) to x-tick labeled b. 3. Draw rectangle using vertical arrow for a and horizontal arrow for b as sides. Label signed number of unit squares, c, rectangle covers. 4. A rectangle in Quadrant I or Quadrant III has a positive signed number of unit squares. A rectangle in Quadrant II or Quadrant IV has a negative signed number of unit squares.

Lesson 4: Narrate algebra using TELL.

Example 4.1: Narrate row 1 in the STEP table from Example 3.1.

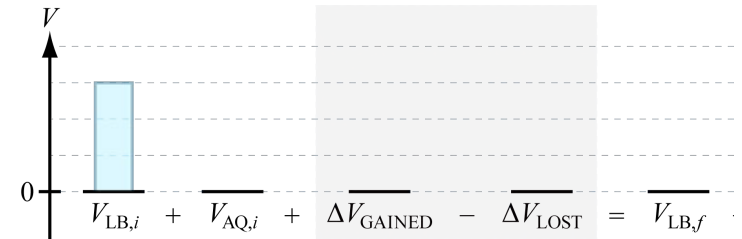
	Style	Tool	Evidence	Linked variables and statements	Loop
1.	Algebraic	By the Addition Property of Equality, $a = b$ implies $a + c = b + c$.	$4x - 2$ plays the role of a , and 98 plays the role of b . Let 2 play the role of c .	So, $4x - 2 = 98$ implies $4x - 2 + 2 = 98 + 2$.	(Not needed)
2.	Natural	By the Addition Property of Equality, a common quantity can be added to both sides of a true equation.	So, in the equation setting left side $4x - 2$ equal to right side 98,	we can add 2 to both sides to get $4x - 2 + 2 = 98 + 2$.	(Not needed)
3.	Brief (merged parts)	By the Addition Property of Equality, a common quantity, say 2, can be added to both sides of a true equation, say $4x - 2 = 98$. So $4x - 2 + 2 = 98 + 2$.			(Not needed)
4.	Even shorter (omit name of tool)	A common quantity, say 2, can be added to both sides of a true equation, say $4x - 2 = 98$. So $4x - 2 + 2 = 98 + 2$.			(Not needed)

Loop: continue linking variables and statements until you reach a sought conclusion.

Lesson 5: Perform spider-web thinking (track quantities and governing and constitutive relationships that constrain them) using **MAP**.

Example 5.1: Make a MAP for the following problem:

A rectangular lunchbox full of water is emptied completely into an initially empty rectangular aquarium that's longer, wider, and taller than the lunchbox. Is the height of the water in the aquarium greater than, less than, or the same as the height of the water initially in the lunchbox?



	Engineer's jargon	Name	Notation	Preferred layout
Managers	Governing relationships among properties belonging to various objects in the system or to interactions with the environment	1. Volume conservation for incompressible fluid	$V_{TOT,BEGIN} + \Delta V_{GAINED} - \Delta V_{LOST} = V_{TOT,END}$	Near center/top Horizontal
Associated Personnel	Constitutive relationships among properties belonging to an individual object within a system or to an interaction with the environment	2. Initial volume of rectangular parcel of water in lunchbox	$V_{LB,i} = \ell_{LB,i} \cdot w_{LB,i} \cdot h_{LB,i}$	Near bottom/edges Vertical
		3. Initial volume of rectangular parcel of water in aquarium	$V_{AQ,i} = \ell_{AQ,i} \cdot w_{AQ,i} \cdot h_{AQ,i}$	
		4. Final volume of rectangular parcel of water in lunchbox	$V_{LB,f} = \ell_{LB,f} \cdot w_{LB,f} \cdot h_{LB,f}$	
		5. Final volume of rectangular parcel of water in aquarium	$V_{AQ,f} = \ell_{AQ,f} \cdot w_{AQ,f} \cdot h_{AQ,f}$	

$$\begin{array}{c}
 \boxed{V_{TOT,BEGIN}} \\
 = \\
 \boxed{V_{LB,i}} + \boxed{V_{AQ,i}} \\
 = \\
 \boxed{\ell_{LB,i}} \cdot \boxed{w_{LB,i}} \cdot \boxed{h_{LB,i}} \\
 + \\
 \boxed{\ell_{AQ,i}} \cdot \boxed{w_{AQ,i}} \cdot \boxed{h_{AQ,i}} \\
 + \\
 \boxed{\Delta V_{GAINED}} - \boxed{\Delta V_{LOST}} = \\
 \boxed{V_{TOT,END}} \\
 = \\
 \boxed{V_{LB,f}} + \boxed{V_{AQ,f}} \\
 = \\
 \boxed{\ell_{LB,f}} \cdot \boxed{w_{LB,f}} \cdot \boxed{h_{LB,f}} \\
 + \\
 \boxed{\ell_{AQ,f}} \cdot \boxed{w_{AQ,f}} \cdot \boxed{h_{AQ,f}}
 \end{array}$$

Read **Manager** relationships horizontally. Read **Associated Personnel** relationships vertically. When investigating a value, consider the possibly multiple relationships that constrain the value, just like when you consider the multiple clues that a single letter must sometimes satisfy in a crossword puzzle.

Lesson 6: Narrate physical reasoning using TELL.

Example 6.1: Narrate an explanation for Example 5.1.

	Style	Tool	Evidence	Linked variables and statements	Loop
1.	Default: Medium-long	The volume of water initially in the lunchbox (water's initial product of length, width, and height) converts fully into the volume of the water finally in the aquarium (water's final product of length, width, and height).	The aquarium's greater length and width mean that, at the end, the water at the bottom of the aquarium has greater length and width, respectively, than the length and width of the water initially in the lunchbox.	So, the water has less height in the aquarium.	(Not needed)
2.	How to lengthen the medium-long version (typically not advised)	State each equation and name each quantity in Example 5.1's MAP.	You'd then need to explicitly enumerate each quantity in the MAP that equals 0.		
3.	Short		The water has a greater length and width when at the bottom of the aquarium than when filling the lunchbox.	So, the water has less height in the aquarium.	