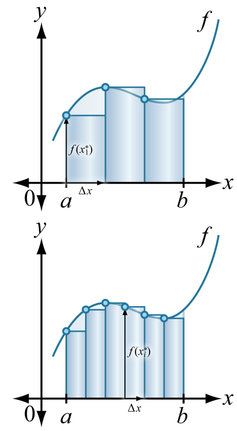
Improper integrals

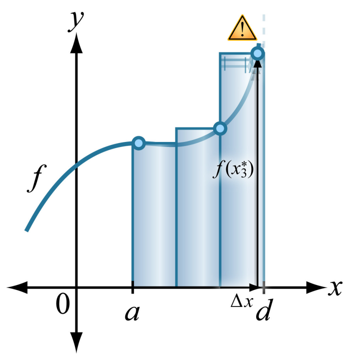
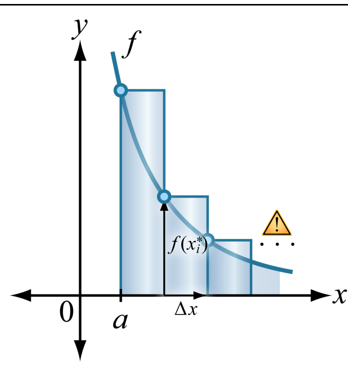
We defined the definite integral of a function f that is **continuous** on $[a, b]$ in terms of a limit of a **Riemann sum**.

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x\,, \qquad \Delta x = \frac{b-a}{n}$$

When we say that the limit above exists,
we mean that the limit exists for any choice of the x_i^*.

The examples below illustrate some ways in which we can write down symbols using the integral sign that, when strictly following the definition of a definite integral, fail to produce a number.



Example undefined symbol	Cartoon	Issue
$\int_a^d \left(\begin{array}{c}\text{formula with}\\ \text{infinite discontinuity}\\ \text{at } x=d\end{array}\right) \mathrm{d}x$		**Requiring the integrand to be continuous** protects us from the following difficulty. Say that we try to choose each x_i^* so as to construct an upper Riemann sum. In this example, no x_i^* can be chosen for the rightmost subregion because the integrand has a (positive-from-the-left) **infinite discontinuity** at $x = d$. Regardless of the number of subintervals in the partition, no upper Riemann sum can be constructed, so the limit of the upper Riemann sum does not exist.
$\int_a^{+\infty} f(x)\,\mathrm{d}x$		**Unbounded interval** of integration **prevents construction of a Riemann sum** using a finite number of rectangular subregions of finite width, so the limit of a Riemann sum does not exist.

Both cartoons illustrate shaded regions that might have finite area (need only finite amount of paint to cover). This intuition conflicts with the fact that the notated definite integrals fail to yield numbers when interpreted directly using the definition of the definite integral in terms of a limit of a Riemann sum. To address this conflict, we declare new meanings for integral symbols like these so that we might be able to assign numbers to such symbols.

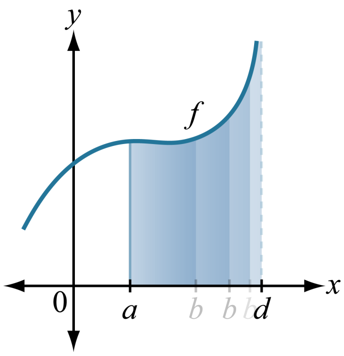
Improper integrals

Unbounded largeness of function (function has infinite discontinuity)

$$\int_a^b f(x)\,\mathrm{d}x$$

is improper when $f(x) \to \pm\infty$ as x approaches one or more values in $[a, b]$.

Vertical asymptote at one end

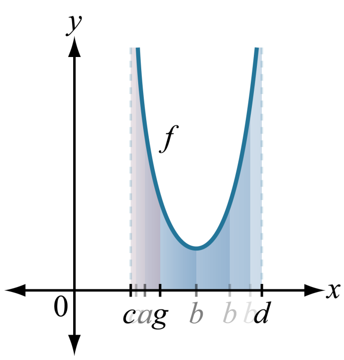


If $f(x)$ becomes unboundedly large as $x \to d^-$,

$$\int_a^d f(x)\,\mathrm{d}x := \lim_{b \to d^-} \int_a^b f(x)\,\mathrm{d}x$$

provided that the limit exists.

Vertical asymptotes at both ends

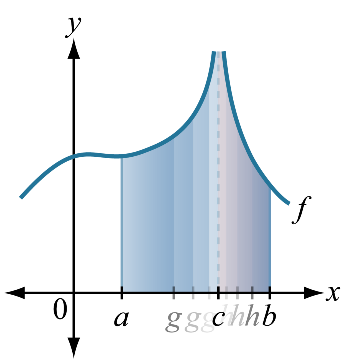


If $f(x)$ becomes unboundedly large as $x \to c^+$ and as $x \to d^-$,

$$\int_c^d f(x)\,\mathrm{d}x := \lim_{a \to c^+} \int_a^g f(x)\,\mathrm{d}x + \lim_{b \to d^-} \int_g^b f(x)\,\mathrm{d}x$$

provided that both limits exist for some g such that $c < g < d$.

Vertical asymptote in interior



If $f(x)$ becomes unboundedly large as $x \to c$, where $a < c < b$,

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{g \to c^-} \int_a^g f(x)\,\mathrm{d}x + \lim_{h \to c^+} \int_h^b f(x)\,\mathrm{d}x$$

provided that both limits exist.

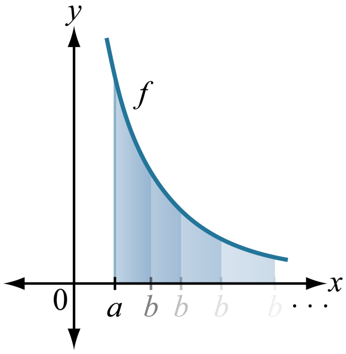
Improper integrals

Interval of integration cannot be bounded by a finite interval

$$\int_a^b f(x)\,\mathrm{d}x$$

is improper when a, b, or both a and b are $\pm\infty$ symbols.

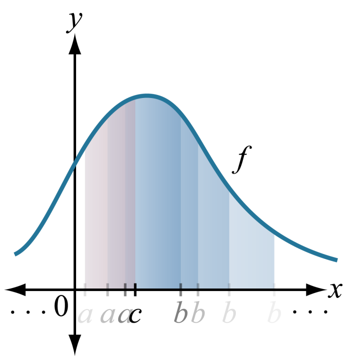
Precisely one boundary expression contains an infinity



$$\int_a^{+\infty} f(x)\,\mathrm{d}x := \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x$$

provided that the limit exists.

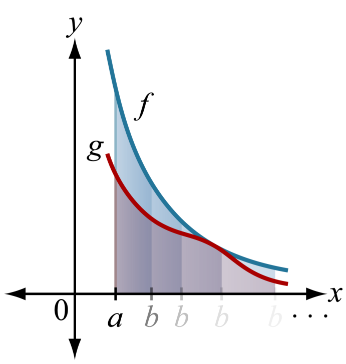
Both boundary expressions contain infinities



$$\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x := \lim_{a\to-\infty}\int_a^c f(x)\,\mathrm{d}x + \lim_{b\to\infty}\int_c^b f(x)\,\mathrm{d}x$$

provided that both limits exist.

Comparison theorem



Hypothesis

1. f and g are continuous functions
2. $f(x) \geq g(x) \geq 0\ \forall x \geq a$

Conclusion

1. $\int_a^{+\infty} f(x)\,\mathrm{d}x \text{ converges} \Rightarrow \int_a^{+\infty} g(x)\,\mathrm{d}x \text{ converges}$
2. $\int_a^{+\infty} g(x)\,\mathrm{d}x \text{ diverges} \Rightarrow \int_a^{+\infty} f(x)\,\mathrm{d}x \text{ diverges}$