LEDGERS FOR JUNIOR-YEAR, COLLEGE, & AI
Carefully read (parse), represent, reason, and write

Dr. David Liao
with use‡ of Claude (Anthropic)

Table of Contents

FOREWARD 2

POINT AND CALL 指差喚呼 2

LESSON 1: DEVELOP AND NARRATE ALGEBRAIC REASONING USING STEP. 3

LESSON 2: LEARN FROM ERRORS USING REDO 4

LESSON 3: DEVELOP AND NARRATE QUALITATIVE REASONING USING STEP. 5

LESSON 4: READ AND WRITE NOTES AND QUESTIONS USING SCAN. 7

LESSON 5: SORT KNOWLEDGE USING AN ADD-SCAN TABLE. 8

LESSON 6: READ A WORD PROBLEM USING SCAN, ORGANIZE BREADTH-TO-DETAIL REASONING USING AN ADD CHART, & SOLVE THE PROBLEM USING STEP. 9

LESSON 7: CARRY OUT INQUIRY (REJECT CANDIDATE MODELS) USING STEAM. 14

LESSON 8: BREAK DOWN PROCEDURALIZED ROUTINES USING STEAM. 15

LESSON 9: KEEP TRACK OF EXCLUDED VALUES AND DOMAIN RESTRICTIONS USING PAINFUL. 16

LESSON 10: IDENTIFY FUNCTION OPERATIONS AND COMPOSITIONS IN CALCULUS USING SCOPE. 17

EXERCISES 17

APPENDIX A: BASIC ALGEBRA TOOLS 19

APPENDIX B: BASIC PRE-ALGEBRA TOOLS 21

APPENDIX C: SPECIFY WHAT EACH SYMBOL IN A STATEMENT MEANS USING DECLARE 22

TOPIC-CORRELATION CHART 23

‡ I, David Liao, used Claude as a research and drafting assistant. I independently verified all text, factual claims, and cited sources against the primary sources. I am solely responsible for the final content. I did not create figures with AI.

CC0 2026 September 09 16h43 EDT

Foreward

Academics

In the United States, school often becomes dramatically more challenging in junior year (11th grade), in college, or both. Students encounter difficulties in **Precalculus**, **Calculus**, and **Physics**.

After tests are handed back and students are guided through solutions, they understand the steps they missed. But how can they come up with the steps *during* tests on their own?

Carefully **read** a good textbook (even if reading isn't assigned) and carry out reasoning in organized **handwritten and hand-drawn work**.

Develop both skills using this packet, which demonstrates how to document thinking in tables to make technical cognition a deliberate craft exposed to adversarial scrutiny, rather than intuition that either arrives or doesn't (and then snowballs into careless guesses).

Artificial Intelligence

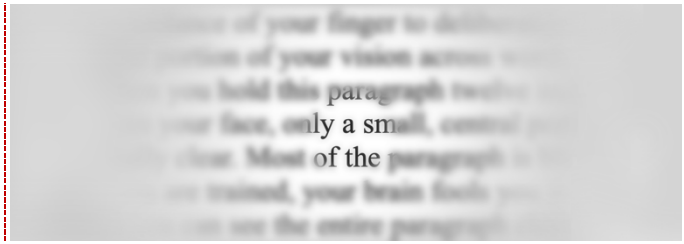
Use this packet to become skilled at technical cognition so you can feed AI high-quality input.

The quality of AI output is highly sensitive to the quality of input. If you ask context-poor questions, you get potentially time-wasting Magic 8 Ball answers that sound persuasive. If you walk AI through thorough context (like a lab's previous journal articles the professor expects incoming grad students to read or a corporation's audited knowledge bases and standard operating procedures) and provide task specifications suitable for commercial contracts, AI acts like a graduate student who asks annoyingly many clarifying questions (of ten such questions, a postdoc would figure out nine without asking) but accomplishes far beyond what a human being can.

Point and call 指差喚呼

Throughout this document, read by pointing and reading aloud 指差喚呼 en.wikipedia.org/wiki/Pointing_and_calling:

> Use the guidance of your finger to deliberately drag the central portion of your vision across words as you read. When you hold this paragraph twelve inches away from your face, only a small, central portion of it is actually clear. Most of the paragraph is blurry. Unless you are trained, your brain fools you into thinking you can see the entire paragraph clearly.



CC0 2026 September 09 16h43 EDT

LESSON 1: Develop and narrate algebraic reasoning using STEP.

Example 1.1: Solve for x. Show each step that can't be accomplished on a basic four-function calculator.

$$4x - 2 = 98$$

	Statement	**T**ool	**E**quivalent **parts** (of statement & tool)	**P**opulated tool / **P**oint(s)
1.	$4x - 2 = 98$	$a = b$ $\updownarrow$ $a + c = b + c$	$a = 4x - 2$ $b = 98$ $c = 2$	$4x - 2 = 98$ $\updownarrow$ $4x - 2 + 2 = 98 + 2$
2.	$4x + 0 = 100$	$a \leftrightarrow a + 0 \leftrightarrow 0 + a$	$a = 4x$	$4x \leftrightarrow 4x + 0 \leftrightarrow 0 + 4x$
3.	$4x = 100$	$a = b$ $\updownarrow$ $\frac{a}{c} = \frac{b}{c}, c \neq 0$	$a = 4x$ $b = 100$ $c = 4$	$4x = 100$ $\updownarrow$ $\frac{4x}{4} = \frac{100}{4}, 4 \neq 0$
4.	$\frac{4x}{4} = 25$	$a \leftrightarrow a \cdot 1 \leftrightarrow 1 \cdot a$	$a = 4$	$4 \leftrightarrow 4 \cdot 1 \leftrightarrow 1 \cdot 4$
5.	$\frac{4x}{4 \cdot 1} = 25$	$\frac{ab}{cd} \leftrightarrow \frac{a}{c}\frac{b}{d}$	$a = 4$ $b = x$ $c = 4$ $d = 1$	$\frac{4x}{4 \cdot 1} \leftrightarrow \frac{4}{4}\frac{x}{1}$
6.	$\frac{4}{4}\frac{x}{1} = 25$	$\frac{a}{a}, a \neq 0 \leftrightarrow 1$	$a = 4$	$\frac{4}{4}, 4 \neq 0 \leftrightarrow 1$
7.	$1 \cdot \frac{x}{1} = 25$	$\frac{a}{1} \leftrightarrow a$	$a = x$	$\frac{x}{1} \leftrightarrow x$
8.	$1 \cdot x = 25$	$a \cdot 1 \leftrightarrow 1 \cdot a \leftrightarrow a$	$a = x$	$x \cdot 1 \leftrightarrow 1 \cdot x \leftrightarrow x$
9.	$x = 25$			

To fill in the **T**ool column, use the basic algebra tools or basic pre-algebra tools from the appendices.

To obtain a two-column Statement-Reason proof in geometry, keep only the **S**tatement and **T**ool columns of a STEP table, move the entries in the **T**ool column down one row, and write "Given" (typically) in the newly vacant entry at the top of the **T**ool column. Homework and tests in Algebra 1 and 2 classes typically only require the work in the **S**tatement column to be shown.

CC0 2026 September 09 16h43 EDT

Example 1.2: Explain your reasoning for row 1 in the STEP table from Example 1.1.

	Style	Skip statement	Tool	Equivalent parts (of statement & tool)	Populated tool / Point(s)
1.	Algebraic	When asked to write an explanation, you often need not rewrite the problem statement or current line of work, which is typically already nearby.	By the Addition Property of Equality, $a = b$ implies $a + c = b + c$.	Regard $4x - 2$ as a, 98 as b, and 2 as c.	So, $4x - 2 = 98$ implies $4x - 2 + 2 = 98 + 2$.
2.	Natural		By the Addition Property of Equality, a common quantity can be added to both sides of a true equation.	So, in the equation setting left side $4x - 2$ equal to right side 98, adding 2 to both sides	gives $4x - 2 + 2 = 98 + 2$.
3.	Brief (merged parts)		By the Addition Property of Equality, a common quantity, say 2, can be added to both sides of a true equation, say $4x - 2 = 98$. So $4x - 2 + 2 = 98 + 2$.		
4.	Even shorter (omit name of tool)		A common quantity, say 2, can be added to both sides of a true equation, say $4x - 2 = 98$. So $4x - 2 + 2 = 98 + 2$.		

LESSON 2: Learn from errors using REDO

Example 2.1: Describe and **correct** the error Charlie made by writing $3(x + 5) = 3x + 5$.

	Right solutions	Correct solutions (try to construct solutions rather than copying solutions from keys)	Statement	Tool	Equivalent parts (of statement & tool)	Populated tool / Point(s)
1.			$3(x + 5)$	$a \cdot (b + c)$ ↕ $a \cdot b + a \cdot c$	$a = 3$ $b = x$ $c = 5$	$3 \cdot (x + 5)$ ↕ $3 \cdot x + 3 \cdot 5$
2.	**Essential knowledge**	Key concepts/skills	According to the Distributive Property, $a \cdot (b + c)$ and $a \cdot b + a \cdot c$ are equivalent.			
3.	**Defective solutions**	Incorrect solutions	$3(x + 5) = 3x + 5$			
4.	**Optimized strategy**	Corrections for misconceptions and procedural errors	To prevent the mistake of multiplying the 3 against only the x rather than against both the x and the 5, write out a row of a **STEP** table using the **Tool** $a \cdot (b + c) \leftrightarrow a \cdot b + a \cdot c$.			

CC0 2026 September 09 16h43 EDT

LESSON 3: Develop and narrate qualitative reasoning using STEP.

Example 3.1: Last year, Cities A and B had median SAT scores of 1200 and 1350, respectively. Charlie's model proposes that a higher city-wide median SAT score last year means a higher city-wide median home price. According to Charlie's model, which city has a higher median home price?

	Statement	**Tool**	**Equivalent parts** (of statement & tool)	**Populated tool / Point(s)**
1.	(Restating a word problem is not usually required).	Charlie's model lower median SAT score → HIGHER median SAT score lower median home price → HIGHER median home price	City A median SAT score 1200 ↔ City B median SAT score 1350 lower median SAT score → HIGHER median SAT score lower median home price → HIGHER median home price	City A median SAT score 1200 ↔ City B median SAT score 1350 lower median SAT score → HIGHER median SAT score lower median home price → HIGHER median home price
2.	City A median SAT score 1200 ↔ City B median SAT score 1350 lower median home price → HIGHER median home price			

According to Charlie's model, City B has a higher median home price.

In fast work, the charts in **Tool**, **Equivalent parts**, and **Populated tool / Point(s)** overlap in the same workspace (no need to create three separate charts).

CC0 2026 September 09 16h43 EDT

Example 3.2: Explain your reasoning for Example 3.1.

	Skip statement	**Tool**	**Equivalent parts** (of statement & tool)	**Populated tool / Point(s)**
1.	(Restating a word problem is not usually required).	By Charlie's model, higher median SAT score means higher median home price.	City B's median SAT score of 1350 was higher than City A's median SAT score of 1200.	So, by Charlie's model, City B has a higher median home price.
2.	When the **Populated tool / Point(s)** entry in the previous row gives just the concise conclusion and answers the question, an additional row restating the conclusion is not needed.			

Example 3.3: Briefly explain your reasoning for Example 3.1.

By Charlie's model, City B has a higher median home price because City B's median SAT score of 1350 was higher.

Procedure for crafting a brief explanation:
1. Omit the **Tool** entry.
2. Reverse the order of the **Populated tool / Point(s)** and **Equivalent parts**.
3. Join the clauses into a single sentence using "because".
4. Omit the "So".

Example 3.4: Create a STEP row for the stock example of classical Indian philosophy, "where there's smoke, there's fire; the hill has smoke, so the hill has fire."

	Statement	**Tool**	**Equivalent parts** (of statement & tool)	**Populated tool / Point(s)**
1.	The hill has smoke. Does the hill have fire?	A location that has smoke is a location that has fire.	Location = hill Smoke = hill's smoke Fire = hill's fire, whose existence is investigated	A hill that has hill's smoke is a hill that has hill's fire, whose existence is investigated.
2.	… hill … has … fire			

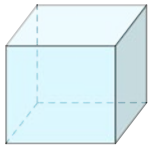
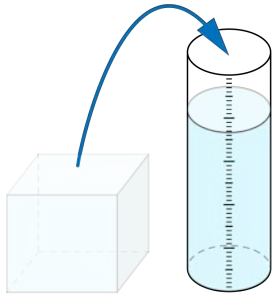
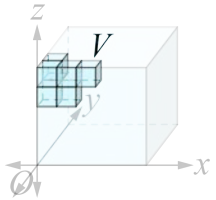
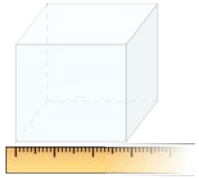
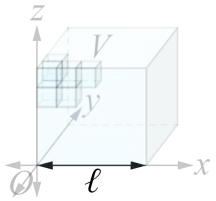
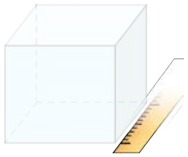
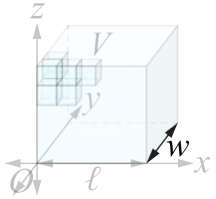
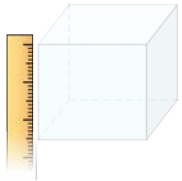
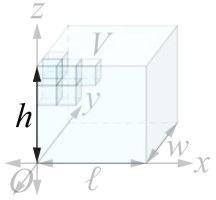
Example 3.5: Create a STEP row with truth tables for symbolic abbreviation of modus ponens, "If p, then q; but p, so q."

	Statement	**Tool**		**Equivalent parts** (of statement & tool)	**Populated tool / Point(s)**	
1.	Given p, is q?	Allow p q	Reject ✗ p not q	p in the tool table = p that's given q in the tool table = q whose existence is investigated	Allow p that's given q whose existence is investigated	Reject ✗ p that's given not q whose existence is investigated
		Allow not p q	Allow not p not q		Allow not p that's given (negated given ✗) q whose existence is investigated	Allow not p that's given (negated given ✗) not q whose existence is investigated
2.	q whose existence is investigated is true.					

CC0 2026 September 09 16h43 EDT

LESSON 4: Read and write notes and questions using SCAN.

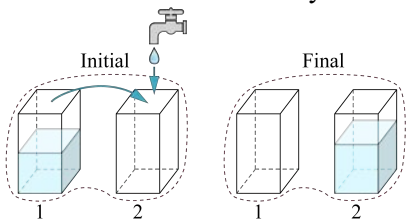
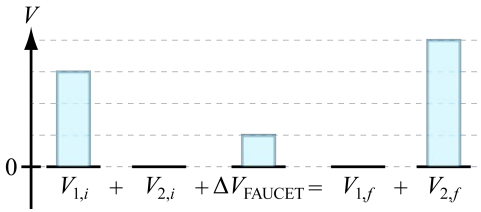
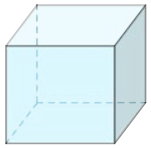
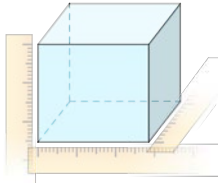
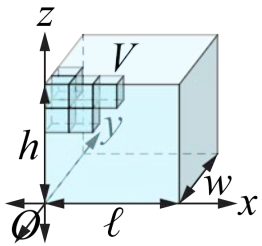
Example 4.1: Translate: "A rectangular parcel of water has a volume equaling the parcel's length times the parcel's width times the parcel's height."

	Short phrase	Cartoon	Axes and plots	Notation
1.	A rectangular parcel of water	Semi-transparent rectangular prism 		
2.	has a volume	Arrow indicating water can be poured from rectangular container into graduated cylinder used to measure volume 	Cartesian axis system implying scale for representative unit cubes within parcel 	Let V = Volume
3.	equaling		Optional: Number line with two arrows pointing from the origin to same tickmark	$V =$
4.	the parcel's length	Optional: Ruler used to measure length 	Double-headed arrow indicating size of an edge parallel to x-axis 	Let ℓ =width $V = \ell$
5.	times		Optional: Area model of multiplication	$V = \ell \cdot$
6.	the parcel's width	Optional: Ruler used to measure width 	Double-headed arrow indicating size of an edge parallel to y-axis 	Let w =width $V = \ell \cdot w$
7.	times		Optional: Area model of multiplication	$V = \ell \cdot w \cdot$
8.	the parcel's height.	Optional: Ruler used to measure height 	Double-headed arrow indicating size of an edge parallel to z-axis 	Let h = height $V = \ell \cdot w \cdot h$

CC0 2026 September 09 16h43 EDT

LESSON 5: Sort knowledge using an ADD-SCAN table.

Example 5.1: Sort knowledge about length, width, height, volume, a formula for the volume of a rectangular prism, and an equation for volume conservation.

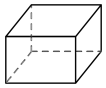
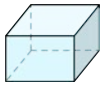
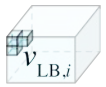
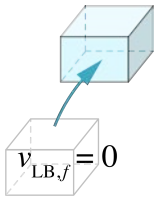
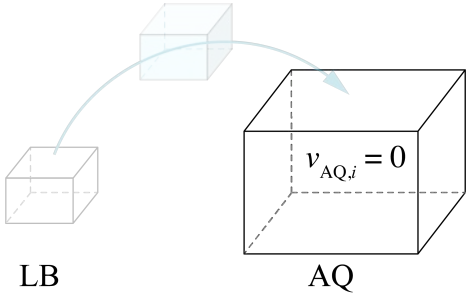
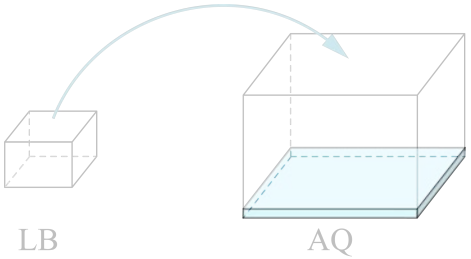
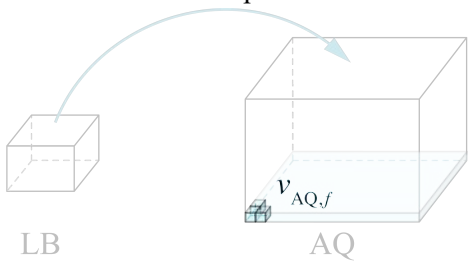
		Short phrase	Cartoon	Axes and plots	Notation	
1.	**Add** up totals	Incompressible fluids obey **conservation of volume:** change in system's total volume is accounted for by exchange with surroundings.	Water volume in final container exceeds water volume in initial container by amount from faucet. Outline two-containers system. 	Volume bar chart 	$V_{\text{TOT},i} + \Delta V_{\text{GAINED}} - \Delta V_{\text{LOST}} = V_{\text{TOT},f}$ $V_{\text{TOT},i} + \Delta V = V_{\text{TOT},f}$	
					$V_{\text{TOT},i}$	Total initial volume
					ΔV_{GAINED}	Volume gained from surroundings
					ΔV_{LOST}	Volume lost to surroundings
					ΔV	Net volume gained from surroundings
					$V_{\text{TOT},f}$	Total final volume
2.	**Details** formulas	A **rectangular prism's volume** equals the product of the prism's length, width, and height.	Cabinet projection of prism 	Highlight some unit cubes (an object's volume is the number of unit cubes that fit in the object).	$V_{\text{RP}} = \ell w h$	
					ℓ	Prism's length
					w	Prism's width
					h	Prism's height
					V_{RP}	Prism's volume
3.	**Data**	Linear dimensions **length**, **width**, and **height** can be measured using a ruler.	Length, width, and height directly measured using rulers 		ℓ	Prism's length
					w	Prism's width
					h	Prism's height
4.		Fluid **volume** is directly measured in a graduated cylinder and indirectly measured by pouring into a graduated cylinder.	Graduated cylinder 	Double-headed arrows label prism's length, width, and height (double-headed arrows indicate endpoint tickmarks of implied coordinate axes).	V	Volume

Add up totals	**Details** formulas	**Data**
Sum of partial amounts equals total. Sum of influences contributes to change. Sum of partial amounts can be changed.	Relate quantities primarily associated with a single object, snapshot, or process.	Physical quantities are values that can be measured using instruments.

CC0 2026 September 09 16h43 EDT

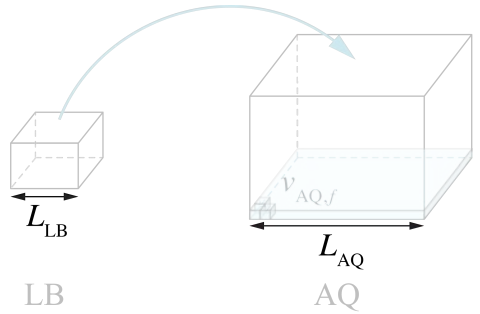
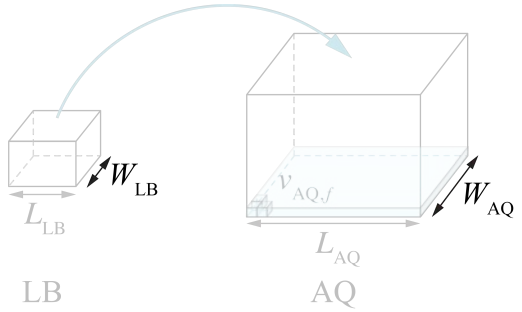
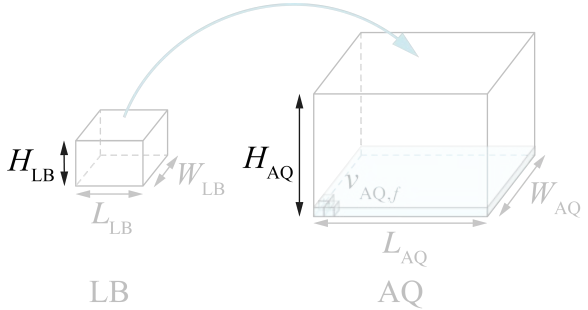
LESSON 6: Read a word problem using SCAN, organize breadth-to-detail reasoning using an ADD chart, & solve the problem using STEP.

Example 6.1: A rectangular lunchbox (LB) full of water is emptied completely into an initially empty rectangular aquarium (AQ) that's longer, wider, and taller than the lunchbox. Is the height of the water in the aquarium greater than, less than, or the same as the height of the water initially in the lunchbox?

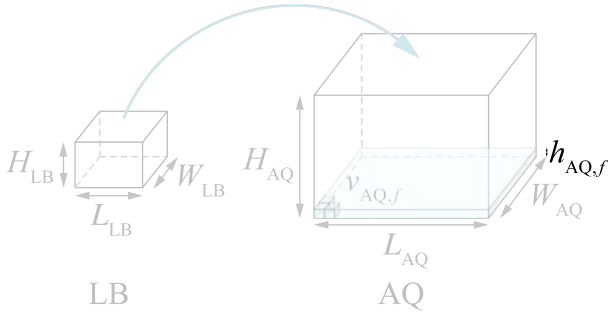
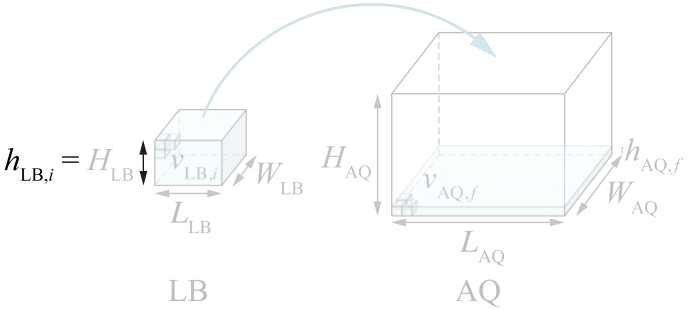
	Short phrase	Cartoon	Axes and plots	Notation
1.	A rectangular lunchbox (LB)	Rectangular prism 		
2.	full of water	Shade rectangular prism blue. 	Highlight some unit cubes. When quantities are necessarily non-negative magnitudes, it's OK to omit Cartesian axes.‡ 	Let $v_{\mathrm{LB},i}$ = lunchbox's initial water volume
3.	is emptied completely	Arrow shows blue parcel of water has moved outside of rectangular container. 		Let $v_{\mathrm{LB},f}$ = lunchbox's final water volume $v_{\mathrm{LB},f} = 0$
4.	into an initially empty rectangular aquarium (AQ)	Lunchbox, parcel of water, and aquarium are represented by rectangular prisms. 		Let $v_{\mathrm{AQ},i}$ = aquarium's initial water volume $v_{\mathrm{AQ},i} = 0$
5.		Move parcel of water into bottom of aquarium. Spread parcel out. 	Highlight some unit cubes in parcel of water in aquarium. 	Let $v_{\mathrm{AQ},f}$ = aquarium's final water volume

‡ By the time the pianist is walking parallel octaves, the imaginary soap bubble used to learn hand shape on day one has popped, but you can still tell that the pianist started from the soap bubble.

CC0 2026 September 09 16h43 EDT

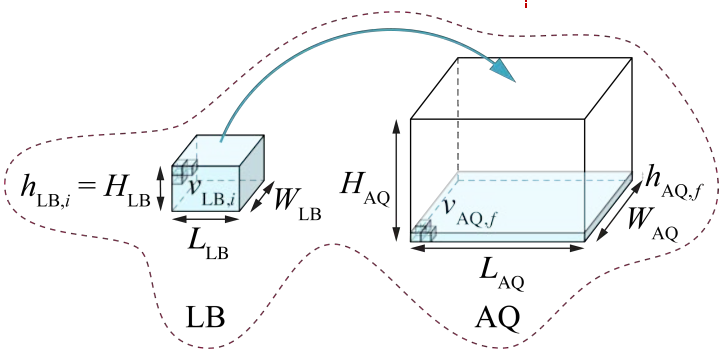
	Short phrase	Cartoon	Axes and plots	Notation
6.	that's longer,		Horizontal double-headed arrow along bottom of aquarium is longer than horizontal double-headed arrow along bottom of lunchbox. 	Let L_{LB} = lunchbox's length Let L_{AQ} = aquarium's length $L_{\mathrm{AQ}} > L_{\mathrm{LB}}$
7.	wider,		Double-headed arrow receding into background along bottom of aquarium is longer than double-headed arrow receding into background along bottom of lunchbox. 	Let W_{LB} = lunchbox's width Let W_{AQ} = aquarium's width $W_{\mathrm{AQ}} > W_{\mathrm{LB}}$
8.	and taller than the lunchbox.		Double-headed arrow along vertical edge of aquarium is taller than double-headed arrow along vertical edge of lunchbox. 	Let H_{LB} = lunchbox's height Let H_{AQ} = aquarium's height $H_{\mathrm{AQ}} > H_{\mathrm{LB}}$

CC0 2026 September 09 16h43 EDT

	Short phrase	Cartoon	Axes and plots	Notation
9.	Is the height of the water in the aquarium		Double-headed arrow along vertical edge of water in aquarium 	Let $h_{AQ,f}$ = aquarium's final water height ?: $h_{AQ,f}$
10.	greater than, less than, or the same as			?: $h_{AQ,f}$ $>, <, =$
11.	the height of the water initially in the lunchbox?		Double-headed arrow along vertical edge of lunchbox is also double-headed arrow along vertical edge of water initially in lunchbox. 	Let $h_{LB,i}$ = lunchbox's initial water height ?: $h_{AQ,f}$ $>, <, =$ $h_{LB,i}$

When making a **SCAN** table becomes reliable, save time and paper by summarizing most **Cartoon** and **Axes and plots** steps in a single illustration, rather than making a multi-page table.

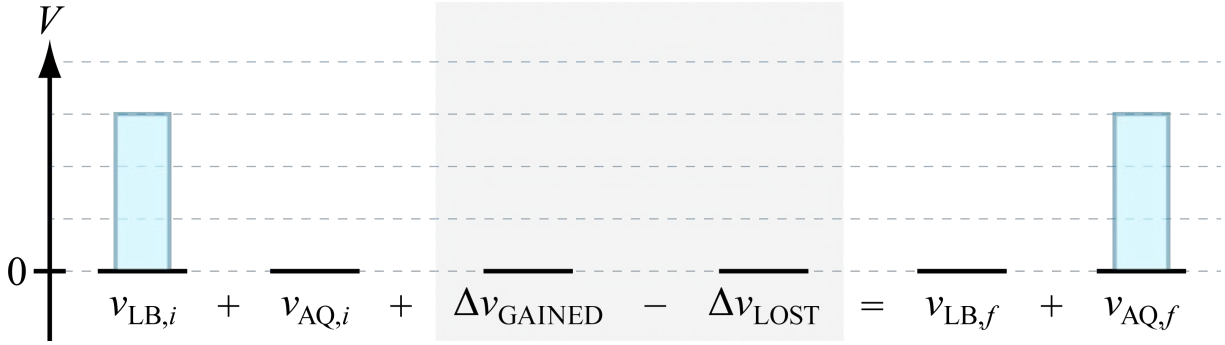
Draw a dashed bubble surrounding the lunchbox and aquarium to indicate that precisely these two containers and the parcel of water the lunchbox initially contains together are the system.



CC0 2026 September 09 16h43 EDT

When creating an **ADD** flowchart, try include a paradigm **Axes and plots** diagram for the main "**Add** up totals" relationship.

In this volume bar chart, we use lowercase v to stand for water volume because, in this problem, we use uppercase letters for properties of the containers (lunchbox and aquarium) and lowercase letters for properties of parcels of water.



Add up totals

$v_{\mathrm{TOT},i}$ + $\Delta v_{\mathrm{GAINED}}$ − Δv_{LOST} = $v_{\mathrm{TOT},f}$

=

$v_{\mathrm{LB},i}$ + $v_{\mathrm{AQ},i}$ = $v_{\mathrm{LB},f}$ + $v_{\mathrm{AQ},f}$

= =

$\ell_{\mathrm{LB},i}$ $\ell_{\mathrm{AQ},i}$ = $\ell_{\mathrm{LB},f}$ $\ell_{\mathrm{AQ},f}$

. .

$w_{\mathrm{LB},i}$ $w_{\mathrm{AQ},i}$ = $w_{\mathrm{LB},f}$ $w_{\mathrm{AQ},f}$

. .

$h_{\mathrm{LB},i}$ $h_{\mathrm{AQ},i}$ = $h_{\mathrm{LB},f}$ $h_{\mathrm{AQ},f}$

Details formulas

Data

CC0 2026 September 09 16h43 EDT

	Style	Skip statement	Tool	Equivalent parts (of statement & tool)	Populated tool / Point(s) (usually just give point(s) in written explanations)
1.	Pre-write	When asked to write an explanation, you often need not rewrite the problem statement or current line of algebraic work, which is typically already nearby.	Draw a **ADD** chart of all relevant relationships.	Fill in given values in your **ADD** chart. If given partial knowledge, note the partial knowledge. For example, you're not given numerical values for ℓ, but you're given that L, and so practically ℓ, increases. So, draw an up arrow ↑ near ℓ.	Use given information you've just filled in to draw conclusions about quantities in your **ADD** chart.
2.	Default: Medium-long		The volume of water initially in the lunchbox (water's initial product of length, width, and height) converts fully into the volume of the water finally in the aquarium (water's final product of length, width, and height).	The aquarium's greater length and width mean that, at the end, the water at the bottom of the aquarium has greater length and width, respectively, than the length and width of the water initially in the lunchbox.	So, the water has less height in the aquarium.
3.	How to lengthen the medium-long version (typically not advised)		State each equation and name each quantity in this example problem's **ADD** chart.	You'd then need to explicitly enumerate each quantity in the **ADD** chart that equals 0.	
4.	Short			The water has a greater length and width when at the bottom of the aquarium than when filling the lunchbox.	So, the water has less height in the aquarium.

CC0 2026 September 09 16h43 EDT

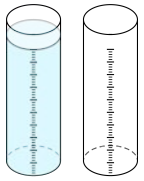
LESSON 7: Carry out inquiry (reject candidate models) using STEAM.

Example 7.1 (Algebraic inquiry): What is $\sqrt{x^2}$? Alice proposes that $\sqrt{x^2} = x$, and Beth propose that $\sqrt{x^2} = |x|$.

	Scenario	Theoretical Expectation		Actual result or Measurement
	x	By Alice's model $\sqrt{x^2} = x$,	By Beth's model $\sqrt{x^2} = \|x\|$,	$\sqrt{x^2}$
1.	-2	-2 😭	2	$\sqrt{(-2)^2} = \sqrt{4} = 2$
2.	-1	-1 😭	1	$\sqrt{(-1)^2} = \sqrt{1} = 1$
3.	0	0	0	$\sqrt{(0)^2} = \sqrt{0} = 0$
4.	1	1	1	$\sqrt{(1)^2} = \sqrt{1} = 1$
5.	2	2	2	$\sqrt{(2)^2} = \sqrt{4} = 2$

In the STEAM table, some of Alice's predictions are **inconsistent** with actual values (when x is negative), so **reject** Alice's model, $\sqrt{x^2} = x$. In the STEAM table, all of Beth's predictions are **consistent** with actual values, so Beth's model $\sqrt{x^2} = |x|$ **remains in play**. The STEAM table alone doesn't allow Beth's model to be "ruled in" since the STEAM table doesn't preclude the possibility of additional values of the shiftable input x resulting in values of $\sqrt{x^2}$ that turn out to fail to equal $|x|$.

Example 7.2 (Physical inquiry): Initially, graduated cylinder A contains some water, and graduated cylinder B does not. Alice proposes that the sum of the volume v_A of water in cylinder A and the volume v_B of water in cylinder B will not change, and Beth proposes that the sum of the volumes of water in the cylinders in a snapshot always equals the volume of water in cylinder A in the same snapshot. Transfer water (only) between cylinders A and B and fill out a STEAM table to challenge Alice's model and Beth's model.



Example data:

	Scenario		Theoretical Expectation		Actual result or Measurement
	v_A (mL)	v_B (mL)	By Alice's model, $v_A + v_B = v_{A,1} + v_{B,1}$,	By Beth's model $v_A + v_B = v_A$	$v_A + v_B$
1.	10.0	0.0	$10.0 + 0.0 = 10.0$	10.0	$10.0 + 0.0 = 10.0$
2.	7.0	3.1	$10.0 + 0.0 = 10.0$	7.0 😭	$7.0 + 3.1 = 10.1$
3.	3.2	6.7	$10.0 + 0.0 = 10.0$	3.2 😭	$3.2 + 6.7 = 9.9$
4.	5.8	4.2	$10.0 + 0.0 = 10.0$	5.8 😭	$5.8 + 4.2 = 10.0$
5.	0.0	10.1	$10.0 + 0.0 = 10.0$	0.0 😭	$0.0 + 10.1 = 10.1$

In the STEAM table, all of Alice's predictions are **consistent** (within the roughly ± 0.1 mL uncertainty of measured water volumes) with actual values, so Alice's model $v_A + v_B = v_{A,1} + v_{B,1}$ **remains in play**. In the STEAM table, some of Beth's predictions are **inconsistent** with actual values (even taking experimental uncertainty into account), so **reject** Beth's model $v_A + v_B = v_A$. The STEAM table alone doesn't allow Alice's model to be "ruled in" since the STEAM table doesn't preclude the possibility of additional values of the shiftable inputs v_A and v_B resulting in values of $v_A + v_B$ that turn out to fail to equal $v_{A,1} + v_{B,1}$.

CC0 2026 September 09 16h43 EDT

LESSON 8: Break down proceduralized routines using STEAM.

Example 8.1 (Significant digits): Use a STEAM table to digest the mini-lesson on counting significant figures/digits in high-school chemistry below.
Rule A: Each non-zero digit is a significant digit.
Rule B: Each zero that is between non-zero digits is a significant digit.
Rule C: Each zero to the left of the first non-zero digit isn't a significant digit.
Rule D: Only when there's a decimal point, each zero to the right of the last non-zero digit is a significant digit.
So, 27 has 2 significant figures, 0.0302 has 3, 60 has 1, and 60. has 2.

	Scenario	**Theoretical Expectation**									**Actual result or Measurement**
		# digits **Rule A** identifies as s. f.		# digits **Rule B** identifies as s. f.		# digits **Rule C** identifies as s. f.		# digits **Rule D** identifies as s. f.				
	Example number from mini-lesson	Non-zero digits	# s. f.	Zeros between non-zero digits	# s. f.	Zeros to left of first non-zero digit	# s. f.	Decimal point?	Zeros to right of last non-zero digit	# s. f. (if there's a decimal point, all, but if there's no decimal point, none)	Total # of s. f. according to rules	Example # of s. f. from mini-lesson
1.	27	2 and 7	2	None	0	None	0	No	None	0	$2+0+0+0=2$	2
2.	0.0302	3 and 2	2	Lone 0 between 3 and 2	1	Two zeros to left of first non-zero digit, 3	0	Yes	None	0	$2+1+0+0=3$	3
3.	60	6	1	None	0	None	0	No	Lone 0 to right of last non-zero digit, 6	0	$1+0+0+0=1$	1
4.	60.	6	1	None	0	None	0	Yes	Lone 0 to right of last non-zero digit, 6	1	$1+0+0+1=2$	2

CC0 2026 September 09 16h43 EDT

LESSON 9: Keep track of excluded values and domain restrictions using PAINFUL.

Example 9.1: Condense $\ln(x) + 2\ln(y)$.

			Forbidden (excluded) values	
			Unstated (implied) exclusions	**L**isted (explicit) exclusions
1.	**P**roblem expression, function, or relationship	$\ln(x) + 2\ln(y)$	The argument of a log must be positive, so $x > 0$ and $y > 0$.	(None given by problem statement)
2.	**A**lgebra **STEP**s (possibly showing only **S**tatements)	$\ln(x) + \ln(y^2)$	(Even in a sadistic high-school course, you can usually skip this section).	
3.	**I**nitial **N**aïve answer	$\ln(xy^2)$	The argument of a log must be positive, so $xy^2 > 0$. x and y must simultaneously be non-zero (or else $xy^2 = 0$). Requiring $y \neq 0$ guarantees that $y^2 > 0$. So, the only remaining requirement to guarantee $xy^2 > 0$ is the requirement that $x > 0$. Summary: $x > 0, y \neq 0$.	Do the exclusions implied by the naïve answer exclude at least those values that were implicitly excluded by and explicitly excluded for the problem expression, function, or equation? The naïve answer's implied exclusion $x > 0$ correctly reproduces the problem's implied requirement that $x > 0$. No additional statement excluding x-values is needed. However, the naïve answer's implied exclusion $y \neq 0$ allows for the possibility that $y < 0$, which was excluded by the problem's implied requirement that $y > 0$. So, add a statement that $y > 0$.
			If the initial naïve answer is a candidate solution set, plug the candidate solutions back into the original problem relationship. Reject candidate statements that cause the problem relationship to be false.	

	Student response	Score	Expression	Exclusion statement
1.	$\ln(x) + 2\ln(y) = \ln(xy^2), y > 0$	**Full** credit ✓	**Correct**	**Correct** and **efficient**
2.	$\ln(x) + 2\ln(y) = \ln(xy^2), x > 0, y > 0$	Substantial **partial** credit ✗	**Correct**	**Correct**, but **inefficient** (stating $x > 0$ is unnecessary)
3.	$\ln(x) + 2\ln(y) = \ln(xy^2)$	Less **partial** credit ✗	**Correct**	**Incorrect**

For schools that enjoy playful torture, include the phrase "assume all variables are positive" in all problem statements for homework and class examples and never explicitly discuss the **PAINFUL** table above in class. Then, omit the phrase "assume all variables are positive" from exams.

CC0 2026 September 09 16h43 EDT

LESSON 10: Identify function operations and compositions in calculus using SCOPE.

			Example 10.1: Identify a candidate limit identity to apply to $\lim_{x\to 2} x^2 + [h(x)]^3$	**Example 10.2:** Identify a candidate differentiation rule to apply to $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x^3)$
1.	**Statement**	**Copy** a line of interest from the current statement.	$\lim_{x\to 2} x^2 + [h(x)]^3$	$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x^3)$
2.	**Container**	Make sure **brackets** enclose the function the limit, derivative, or integral acts on.	$\lim_{x\to 2}\{x^2 + [h(x)]^3\}$	$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(x^3)]$
3.	**Order** of operations	Pick an **example** numerical **value of x**, say 2.4. Pretend you have a **basic scientific calculator** (TI-30Xa) that makes you type each operation separately. List the **sequence of button pushes** needed to evaluate the expression in the container brackets, presuming that evaluation is possible.	1. 2.4 [y^x] 2 [=]. 2. Write down 2.4^2, which is 5.76. 3. 2.4 [h button] [y^x] 3 [=]. 4. Write down $[h(2.4)]^3$, which is 🙂. 5. 5.76 [+] 🙂 [=].	1. Press [DRG] as needed to set calculator into radian mode. 2. 2.4 [y^x] 3 [=]. 3. [sin].
4.	**Punchline**	Identify **the last operation** (or cluster of operations). Highlight and/or make bold the corresponding symbo(s)l in an updated version of the statement.	$\lim_{x\to 2}\{x^2 + [h(x)]^3\}$	$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(x^3)]$
5.	**Expressions**	Identify the **value**(s) the **last operation** (or cluster of operations) acted on. Make sure each such value is clearly enclosed in parentheses in a new copy of the statement (draw parentheses if needed).	$\lim_{x\to 2}\{(x^2) + ([h(x)]^3)\}$	$\frac{\mathrm{d}}{\mathrm{d}x}[\sin(x^3)]$
6.			Highlighting makes it easier to consider using the limit-of-a-sum identity $\lim_{x\to c}\{[f(x)] + [g(x)]\}$ ↕ (fine print omitted) $\left\{\lim_{x\to c}[f(x)]\right\} + \left\{\lim_{x\to c}[g(x)]\right\}$ with $c = 2$, $f(x) = x^2$, and $g(x) = [h(x)]^3$.	Highlighting makes it easier to consider using the chain rule $\frac{\mathrm{d}}{\mathrm{d}x}[f(g(x)]$ ↕ (fine print omitted) $\frac{\mathrm{d}}{\mathrm{d}y}[f(y)]\|_{y=g(x)}\frac{\mathrm{d}}{\mathrm{d}x}[g(x)]$ with $f(x) = \sin(x)$ and $g(x) = x^3$.

CC0 2026 September 09 16h43 EDT

Exercises

Lesson 1: STEP

Exercise 1.1: Solve for x in each of the following equations.
(a) $-5x + 2 = 12$
(b) $3(x - 1) = 15$
(c) $2x + 5x = 49$
(d) $4x + 5 = 9x$
(e) $x^2 = 9$
(f) $x^4 + 4 = 5x^2$

Exercise 1.2: Explain your reasoning for each step in the STEP tables from Exercise 1.1.

Lesson 3: STEP

Exercise 3.1: If higher frequency for a sound wave means higher pitch for the corresponding sound that's heard, which of frequencies 440 Hz and 880 Hz corresponds to a higher pitch?

Exercise 3.2: Explain your reasoning for Exercise 3.1.

Exercise 3.3: Create a STEP table for the winning argument made by Elle Woods (Reese Witherspoon) in *Legally Blonde* (youtu.be/GSu7BGbyJqc).

Lesson 4: SCAN

Exercise 4.1: Translate: "An object that travels for a duration of time t at constant speed v travels a distance d equal to the product of the duration and speed."

Exercise 4.2: Translate: "The initial value of a quantity plus the change in the value of that quantity equals the quantity's final value. The ratio of the change in price of a basket of goods through a year to the price of that basket of goods at the start of the year is the annual inflation rate measured using that basket."

Lesson 5: ADD-SCAN table

Exercise 5.1: Sort knowledge about ingredient weight in a container, container weight, percentage by weight, total weight in terms of container weights, and total weight of an ingredient in terms of weights of that ingredient in containers using a **ADD-SCAN** table.

Exercise 5.2: Sort knowledge about distance, speed, duration, a distance formula, total journey distance in terms of leg distances, and total journey duration in terms of leg durations using a **ADD-SCAN** table.

Lesson 6: Solving word problems using SCAN, ADD, and STEP

Exercise 6.1: A three-pound bag of trail mix contains 30% raisins by weight. A five-pound bag of trail mix contains 20% raisins by weight. Of the combined trail mix from the two bags, what percent by weight is raisins?

Exercise 6.2: Narrate your reasoning for exercise 6.1.

Exercise 6.3: The constant speed with which a boat travels from City A to City B is $5\frac{\text{miles}}{\text{hour}}$ less than the constant speed with which the boat travels from City B to City A. The length of the water route between Cities A and B is 50 miles, and the round-trip travel time (not including time spent docked) is 6 hours. What is the boat's speed during each leg of its trip?

Exercise 6.4: Narrate your reasoning for exercise 6.3.

Lesson 7: STEAM

Exercise 7.1: Test the proposal that $\frac{x}{x} = 1$ and another proposed formula of your choosing for $\frac{x}{x}$.

Exercise 7.2: Test the proposal that $\left(\sqrt{x}\right)^2 = x$ and another proposed formula of your choosing for $\left(\sqrt{x}\right)^2$.

Exercise 7.3: Create a STEAM table for the differential diagnosis Mona Lisa Vito (Marisa Tomei) and Vinny Gambini (Joe Pesci) walk through in *My Cousin Vinny* (youtu.be/EvYQTvCVYz0?t=84).

Lesson 9: PAINFUL

Exercise 9.1: Simplify the expression.
(a) $\frac{x^2}{x}$
(b) $\left(\sqrt{2x - 1}\right)^2$

Exercise 9.2: Solve for x.
$$\log_{10}(x^2 - 6) = \log_{10}(x)$$

Exercise 9.3: Perform the indicated function operation.
(a) $f(x) = \sqrt{x - 2} + 3$
$g(x) = x - \sqrt{x - 2} - 3$
Find $f(x) + g(x)$

(b) $f(x) = \frac{x^2 - 6x - 9}{x + 2}$
$g(x) = \frac{x - 3}{x^2 - 4}$
Find $\frac{f(x)}{g(x)}$

CC0 2026 September 09 16h43 EDT

Appendix A: Basic algebra tools

For exercises, see corresponding "EA" sections in Marecek *et al.*, *Elementary Algebra 2e*, available for free through a CC BY license at OpenStax: openstax.org/details/books/elementary-algebra-2e

EA	Name	Tool(s)
1.9	**A.** Division by zero is undefined	Any denominator or factor in denominator = 0 ↓ Discard template
1.5	**B.** Product of fractions	$\frac{a}{b}\cdot\frac{c}{d}\leftrightarrow\frac{ac}{bd}$
1.5	**C.** Canceling factors in numerator & denominator	$\frac{a\cdot c}{b\cdot c}\leftrightarrow\frac{a}{b}$ $\quad$ $\frac{c\cdot a}{c\cdot b}\leftrightarrow\frac{a}{b}$
1.5	**D.** Quotient of fractions	$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)}\leftrightarrow\frac{a}{b}\cdot\frac{d}{c}$
1.9	**E.** Commutative Property of Addition	$a+b$ ↕ $b+a$
1.9	**F.** Commutative Property of Multiplication	$a\cdot b$ ↕ $b\cdot a$
1.9	**G.** Associative Property of Addition	$a+b+c$ ↕ $(a+b)+c$ ↕ $a+(b+c)$
1.9	**H.** Associative Property of Multiplication	$a\cdot b\cdot c$ ↕ $(a\cdot b)\cdot c$ ↕ $a\cdot(b\cdot c)$

EA	Name	Tool(s)
1.9	**I.** Distributive Property	$a\cdot(b+c)$ ↕ $a\cdot b+a\cdot c$ $\quad$ $(b+c)\cdot a$ ↕ $b\cdot a+c\cdot a$
		$a\cdot(b-c)$ ↕ $a\cdot b-a\cdot c$ $\quad$ $(b-c)\cdot a$ ↕ $b\cdot a-c\cdot a$
		$(a+b)\cdot(c+d)$ ↕

	c	d
a	$a\cdot c$	$a\cdot d$
b	$b\cdot c$	$b\cdot d$

$a\cdot c+b\cdot c+a\cdot d+b\cdot d$

1.9	**J.** Identity Property of Addition	$a+0\leftrightarrow 0+a\leftrightarrow a$
1.9	**K.** Identity Property of Multiplication	$a\cdot 1\leftrightarrow 1\cdot a\leftrightarrow a$
1.9	**L.** Identity Property of Division	$\frac{a}{1}\leftrightarrow a$
1.9	**M.** Adding additive inverse is equivalent to subtracting	$a+(-b)$ ↕ $a-b$
1.9	**N.** Inverse Property of Addition	$a+(-a)\leftrightarrow a-a\leftrightarrow 0$
1.5 & 1.9	**O.** Inverse Property of Multiplication (and the related One Property of Division)	$a\cdot\frac{1}{a}, a\neq 0\leftrightarrow\frac{a}{a}, a\neq 0\leftrightarrow 1$

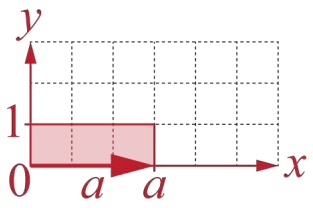
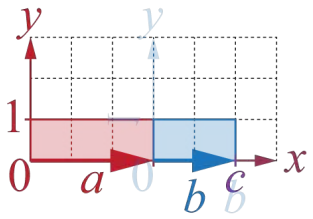
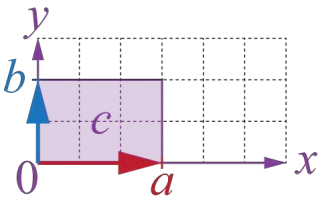
CC0 2026 September 09 16h43 EDT

EA	Name	Tool(s)	
1.9	**P.** Multiplication by zero	$a \cdot 0 \leftrightarrow 0 \cdot a \leftrightarrow 0$	
1.9	**Q.** Division involving zero	$\frac{0}{a} \leftrightarrow 0$	See "Division by zero is undefined"
2.1	**R.** Addition Property of Equality	$a = b$ $\downarrow$ $a + c = b + c$	
2.1	**S.** Subtraction Property of Equality	$a = b$ $\downarrow$ $a - c = b - c$	
2.2	**T.** Multiplication Property of Equality	$a = b$ $\downarrow$ $a \cdot c = b \cdot c$	
2.2	**U.** Division Property of Equality	$a = b$ $\downarrow$ $\frac{a}{c} = \frac{b}{c}$	

EA	Name	Tool(s)	
6.2	**V.** Exponent notation	$a^2 \leftrightarrow a \cdot a$	$a^n \leftrightarrow \underbrace{a \cdot a \cdots a}_{n \text{ copies}}$
7.6	**W.** Zero-Product Property	$a \cdot b = 0$ $\downarrow$ $a = 0$ or $b = 0$	
9.1/9.7	**X.** Root notation	$\sqrt[n]{a} = b$, even n $\updownarrow$ $b^n = a, b \geq 0$, even n	$\sqrt[n]{a} = b$, odd n $\updownarrow$ $b^n = a$, odd n
9.8*	**Y.** A correction to the Power Property	$\sqrt{a^2} \leftrightarrow \lvert a \rvert$	
10.3	**Z.** Quadratic Formula	$ax^2 + bx + c = 0$ $\updownarrow$ $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	

CC0 2026 September 09 16h43 EDT

Appendix B: Basic pre-algebra tools

	Name	Numeric example	Tool		Gridding steps
			Algebraic	Gridded	
A.	**(Real) number**	3	a		1. Draw horizontal arrow from origin (0) to x-tick labeled a. Label arrow a. 2. Regard horizontal arrow as bottom edge of a rectangle of height 1. 3. Horizontal arrow and rectangle represent the number a individually and together.
B.	**Addition**	$3 + 2 = 5$	$a + b = c$		1. Represent the number a. 2. Regard arrowhead of arrow for a as origin of a new axis system. 3. Use new axis system to represent the number b. 4. Use original axes to read off x-tick, c, of arrowhead you just drew and complete sketch of the number c.
C.	**Subtraction**	$2 = 5 - 3$	$b = c - a$		
E.	**Multiplication**	$(3)(2) = 6$	$(a)(b) = c$		1. Represent number a. Erase rectangle. 2. Draw vertical arrow from origin (0) to y-tick labeled b. 3. Draw rectangle using horizontal arrow for a and vertical arrow for b as sides. Label signed number of unit squares, c, rectangle covers. 4. A rectangle in Quadrant I or Quadrant III has a positive signed number of unit squares. A rectangle in Quadrant II or Quadrant IV has a negative signed number of unit squares.
F.	**Division**	$2 = \frac{6}{3}$	$b = \frac{c}{a}, a \neq 0$		

CC0 2026 September 09 16h43 EDT

Appendix C: Specify what each symbol in a statement means using DECLARE

Use this appendix if geometric figures and their measurements are getting confused in geometry. Enumerate scenes, objects, and abilities and properties of objects, to unambiguously specify meanings of symbols in shorthand formulas.

Example C (geometric formula): Suppose you'd like to write out the meaning of the following statement.

The length AB of segment $\overline{AB}$ with endpoints $A(x_1, y_1)$ and $B(x_2, y_2)$ equals $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Use a **DECLARE** table to clarify what each object label and algebraic symbol means.

	Date/Time	Environment	Category of thing	Label	Ability	Readout	Algebraic Expression
1.	Now	Plane					
2.	Now	Plane	Point	A			
3.	Now	Plane	Point	A		x-coordinate	x_1
4.	Now	Plane	Point	A		y-coordinate	y_1
5.	Now	Plane	Point	B			
6.	Now	Plane	Point	B		x-coordinate	x_2
7.	Now	Plane	Point	B		y-coordinate	y_2
8.	Now	Plane	Segment	$\overline{AB}$			
9.	Now	Plane	Segment	$\overline{AB}$		length	AB

Write a **long-hand** version that includes context reminders near each named object, action, and quantity.

$$\text{Now. Plane. } \overline{AB}.\text{length} = \sqrt{(\text{Now. Plane. } B.x\text{coord} - \text{Now. Plane. } A.x\text{coord})^2 + (\text{Now. Plane. } B.y\text{coord} - \text{Now. Plane. } A.y\text{coord})^2}$$

Then write a **shorthand** version that suppresses context.

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

CC0 2026 September 09 16h43 EDT

Topic-correlation chart

Consuming & applying knowledge

Curriculum	Tasks	Skill	Scaffold	References
Algebra Geometry AP® Calculus ACT® Science AP® Chemistry AP® Physics English AP® Seminar Forensics Speech & Debate PARCC NJSLA	Solve for x. Prove. Explain. Briefly explain. Argue.	**Develop & narrate arguments**	**STEP**	**Warrant-licensed & related arguments** Major premise 4th century BCE iep.utm.edu/aristotle-logic/#H9 法 late 4th-mid 3rd century BCE plato.stanford.edu/entries/mohist-canons/#ArguLogi Stoic first indemonstrable 3rd century BCE britannica.com/topic/history-of-logic/The-Megarians-and-the-Stoics Vyāpti developed 2nd-7th century CE iep.utm.edu/nyaya/#SSH1bii & jstage.jst.go.jp/article/ibk1952/35/1/35_1_476/_pdf Kubrā ~10th century CE plato.stanford.edu/entries/ibn-sina-logic/ Toulmin 2003 (1958) doi.org/10.1017/CBO9780511840005 IRAC 1961 jle.aals.org/home/vol72/iss3/3/ Luketic, Elle wins! *Legally Blonde* 2001 youtu.be/GSu7BGbyJqc McNeill *et al.* CER 2006 doi.org/10.1207/s15327809jls1502_1
				Reasoning chains Speirs *et al.* 2016 doi.org/10.1119/perc.2016.pr.077
	Corrections	**Learn from errors**	**REDO**	**Assessment corrections** Henderson & Harper 2009 doi.org/10.1119/1.3264589
	Word problems Read textbooks	**Read** & **write** notes & questions	**SCAN**	**Multiple representations** 析理以辭，解體用圖 3rd century CE ctext.org/wiki.pl?if=gb&chapter=437992 & mathshistory.st-andrews.ac.uk/Biographies/Liu_Hui/ Larkin *et al.* 1980 doi.org/10.1126/science.208.4450.1335 van Heuvelen 1991 doi.org/10.1119/1.16668 Hughes-Hallett 2010 & Chang *et al.* 2015 files.eric.ed.gov/fulltext/ED557199.pdf
				Idea isn't name Arons 1984 doi.org/10.1119/1.2341444
	Organize formulary	**Sort** knowledge	**ADD**	**Novice & expert problem-solving, schemata** Chi *et al.* 1981 doi.org/10.1207/s15516709cog0502_2
	Word problems	**breadth-to-detail reasoning**		**Simultaneously diagrammed governing & constitutive relationships** Rosengrant 2011 doi.org/10.1119/1.3527754

Producing knowledge

Curriculum	Tasks	Skill	Scaffold	References
Algebra Geometry AP® Calculus ACT® Science AP® Chemistry AP® Physics	Testing experiments	**Inquire** (reject candidate models)	**STEAM**	**Inquiry-based learning** Etkina *et al.* islephysics.net
				Explicitly stating prior models Schneps & Sadler 1989 lweb.cfa.harvard.edu/smgdvl/pu/pu.mp4
				Differential diagnosis Lynn, The defense is wrong! *My Cousin Vinny* 1992 youtu.be/EvYQTvCVYz0?t=84
				Refutability Popper plato.stanford.edu/entries/popper/

CC0 2026 September 09 16h43 EDT

Attending to detail

Curriculum	Tasks	Skill	Scaffold	References
Hard Honors Precalculus	Solve for x. Simplify.	Note excluded values & **domain restrictions**	**PAINFUL**	**Implied domain** Jones people.richland.edu/james/lecture/m116/functions/functions.html
AP® Calculus	Evaluate limit, derivative, or integral.	Identify function operations & compositions	**SCOPE**	**Outermost operation is last evaluation step** Dawkins tutorial.math.lamar.edu/Classes/CalcI/ChainRule.aspx
Geometry AP® Physics	Take notes. Prove.	**Distinguish** geometric **figures** & **measures**	**DECLARE**	**Explicit ontologies** Object Constraint Language 2014 omg.org/spec/OCL/2.4/PDF **Objects vs. measures** Perrin-Glorian 1990 irem.univ-grenoble-alpes.fr/medias/fichier/24x1_1570438041435-pdf

CC0 2026 September 09 16h43 EDT